

# The Conway-Miyamoto correspondences for the Fischer 3-transposition groups

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## Abstract

In this paper, we present a general construction of 3-transposition groups as automorphism groups of vertex operator algebras. Applying to the moonshine vertex operator algebra, we establish the Conway-Miyamoto correspondences between Fischer 3-transposition groups  $\text{Fi}_{23}$  and  $\text{Fi}_{22}$  and  $c = 25/28$  and  $c = 11/12$  Virasoro vectors of subalgebras of the moonshine vertex operator algebra.

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# 1 Introduction

The discovery of sporadic finite simple groups is one of the most mysterious steps in the classification of finite simple groups. The origin of these sporadic groups is still missing so far. The most natural way to view a group is to realize it as the automorphism group of some canonical objects. Like the Tits building of a finite simple group of Lie type, some sporadic finite simple groups can be realized as automorphism groups of certain structures. For instance, the binary Golay code is the natural object for the Mathieu groups; the Leech lattice is the one for the Conway groups; and the moonshine vertex operator algebra is the one for the Monster [FLM88]. There is no doubt that the Golay code and the Leech lattice are the canonical objects for the Mathieu groups and the Conway groups. Many important properties about the Mathieu groups and the Conway groups (such as the orders) can be determined directly by analyzing the corresponding objects, and we expect a similar story for the Monster. In its relation to the solution of the moonshine conjecture by Borcherds [B86], the moonshine VOA would be the most canonical object for the Monster. However, it is not easy to handle the moonshine VOA directly since it is infinite dimensional and has a very complicated, but beautiful, structure. The purpose of this paper is to establish non-trivial interconnections between the Fischer 3-transposition groups and certain vertex operator subalgebras of the moonshine vertex operator algebra and to ensure that vertex operator algebras are really canonical and natural objects for some of sporadic groups in the Monster family.

Conway [C85] reconstructed a slightly modified version of the original Griess algebra [G98a] of dimension  $1 + 196883$  and described a 2A-involution of the Monster in terms of the adjoint action of an idempotent called the transposition axis. Namely, given a 2A-involution  $t$  of the Monster  $\mathbb{M}$ , the  $C_{\mathbb{M}}(t)$ -invariants of the Griess algebra forms a two dimensional subalgebra spanned by two mutually orthogonal idempotents and the involution  $t$  can be reconstructed by the eigenspace decomposition of the shorter  $C_{\mathbb{M}}(t)$ -invariant idempotent. The Conway's modified Griess algebra is naturally realized as the weight two subspace of the moonshine VOA  $V^\natural$  [FLM88]. Miyamoto [Mi96] showed that the Conway's axis corresponds to a  $c = 1/2$  Virasoro vector, called an *Ising vector*, of  $V^\natural$  and he defined to each Ising vector an involution of  $V^\natural$  based on the fusion rules of the Virasoro VOA, which may be viewed as a converse of the Conway's axis in a sense, and it turns out that there exists a one-to-one correspondence between the 2A-elements of the Monster and the Ising vectors of the moonshine VOA (cf. [C85, Mi96, Ma01, H10]).

For a non-negative integer  $n$ , let

$$c_n = 1 - \frac{6}{(n+2)(n+3)}$$

be the central charge of the unitary series of the Virasoro algebra. If a VOA  $V$  contains a simple  $c = c_n$  Virasoro vector  $e$ , then the linear map  $\tau_e = (-1)^{4(n+2)(n+3)o(e)}$  is well-defined, where  $o(e) = e_{(1)}$  is the zero-mode of  $e$ , and gives rise to an automorphism of  $V$  called the *Miyamoto involution* associated to  $e$  (cf. [Mi96]). The case  $c_1 = 1/2$  in  $V^\natural$  produces a 2A-element of Conway's axis. Similarly, one can define another Miyamoto involution  $\sigma_e$  on a certain sub VOA of  $V$  using a similar method. (See Theorem 2.4 for detail.) When  $\tau_e$  is trivial and  $\sigma_e$  is well-defined on the whole space  $V$ , such a Virasoro vector  $e$  is called of  $\sigma$ -type on  $V$ .

One can use Virasoro vectors other than Ising vectors to define 2A-elements of the Monster. For example, it is shown in [Ma01] that the Miyamoto involutions associated to  $c = c_2$  Virasoro vectors of the moonshine VOA also produce 2A-involutions of the Monster. However, based on Conway's observation on axial vectors, the nature of the 2A-involutions of the Monster is canonically attached to the Ising vectors of the moonshine VOA. Motivated by works of Conway [C85] and Miyamoto [Mi96], we introduce the notion of the Conway-Miyamoto correspondence as follows. Let  $V$  be a VOA,  $G$  a subgroup of  $\text{Aut}(V)$  and  $I$  a conjugacy class of involutions of  $G$ . We define the *Conway-Miyamoto correspondence between involutions  $I$  of  $G$  and  $c = c_n$  Virasoro vectors of  $V$*  by the following conditions:

- (1) For each  $t \in I$ , there exists a unique  $c = c_n$  Virasoro vector in  $V^{C_G(t)}$ .
- (2) If the unique Virasoro vector  $e_t$  is not of  $\sigma$ -type on  $V$ , then  $\tau_{e_t} = t$  on  $V$ .
- (3) If  $e_t$  is of  $\sigma$ -type on  $V$ , then  $\sigma_{e_t} = t$  on  $V$ .

The unique  $c = c_n$  Virasoro vector  $e_t$  of  $V^{C_G(t)}$  is called the *axial vector* associated to  $t \in I$ . Note that the condition (1) is imposed independently of the condition (2). Namely, the axial vector  $e_t$  is uniquely determined by the action of the centralizer  $C_G(t)$  and its central charge. Since  $\sigma_{ge_t} = g\sigma_{e_t}g^{-1}$  if  $e_t$  is of  $\sigma$ -type on  $V$  and  $\tau_{ge_t} = g\tau_{e_t}g^{-1}$  otherwise for  $g \in G$ , the type of Miyamoto involutions of the axial vectors is uniquely chosen by the class  $I$ . We say the Conway-Miyamoto correspondence is *bijective* if for each  $t \in I$  the axial vector  $e_t$  is the unique  $c = c_n$  Virasoro vector producing  $t$  via the Miyamoto involution of specified type.

The most prominent example of Conway-Miyamoto correspondences is given by the 2A-involutions of the Monster and the Ising vectors of the moonshine VOA. The Conway-Miyamoto correspondence between the 2A-elements of the Baby Monster and  $c = c_2$  Virasoro vectors of  $\sigma$ -type of the Baby Monster VOA is established in [HLY12a] and that between the 2C-elements of the largest Fischer 3-transposition group and the  $c = c_4$  Virasoro vectors of  $\sigma$ -type of the Fischer VOA in [HLY12b]. Both of the Baby Monster VOA and the Fischer VOA are commutant subalgebras of the moonshine VOA and these correspondences are derived from the Monster. Once a Conway-Miyamoto correspondence has been established, we can internally describe the corresponding automorphism group based on the VOA structure and study the group as an intrinsic symmetry of the corresponding VOA. In [LM06, LYY07, LYY05, S07], subalgebras generated by two Ising vectors of the moonshine VOA are studied and these subalgebras are used to study McKay's  $E_8$  observation of the Monster. Similarly, McKay's  $E_7$  and  $E_6$  observations are studied using the Baby Monster VOA and the Fischer VOA based on the Conway-Miyamoto correspondences in [HLY12a, HLY12b]. The other successful examples of Conway-Miyamoto correspondences are provided by VOAs related to Matsuo algebras and associated 3-transposition groups [Ma05, HRS15, LY14, R15], where the structures of Fischer spaces are directly encoded into Matsuo algebras which appear as Griess algebras of VOAs. For those VOAs related to (indecomposable) Matsuo algebras it is shown in [Ma05] that the Conway-Miyamoto correspondences are always bijective. As examples, there exist bijective Conway-Miyamoto correspondences between transvections (2E-elements in [ATLAS]) of  $O_{10}^+(2)$  and Ising vectors of  $V_{\sqrt{2}E_8}^+$  (cf. [G98a, Ma05, LSY07]), and between reflections of  $\Omega_8^-(3).2$  and  $c = c_3$  Virasoro vectors of  $\sigma$ -type of  $V_{K_{12}}^{(\mu)}$  (cf. [LY14]), where  $K_{12}$  is the Coxeter-Todd lattice of rank 12.

In this paper, we will establish the Conway-Miyamoto correspondences for the second and the third largest Fischer 3-transposition groups. According to the classification of center-free 3-transposition groups in [As97, CH95, Fi71], there are three Fischer 3-transposition groups  $Fi_{24}$ ,  $Fi_{23}$  and  $Fi_{22}$ .  $Fi_{24}$  is the largest one and the remaining two groups are inductively obtained as the centralizers of a Fischer transposition. We will

show that Fischer transpositions of  $\text{Fi}_{23}$  and  $\text{Fi}_{22}$  correspond to  $c = c_5$  and  $c = c_6$  Virasoro vectors of certain sub VOAs of the moonshine VOA.

To this end, we will consider dihedral subalgebras and their commutant subalgebras in a general VOAs satisfying mild conditions. A subalgebra generated by two Ising vectors is called a dihedral subalgebra. Griess algebras of dihedral subalgebras of OZ-type are classified in [S07] and it turns out that all of them are realizable inside the moonshine VOA (cf. [LM06, LYY07, LYY05]). We will make use of the dihedral subalgebras of 2A, 3A and 6A-types, which are simply called the 2A, 3A and 6A-algebras, respectively. In [HLY12b], the  $W_3$ -algebra at  $c = 4/5$  is used to define the Fischer VOA as the commutant sub VOA of the moonshine VOA. The  $W_3$ -algebra at  $c = 4/5$  is a subalgebra of the 3A-algebra and in this paper we will consider the commutant sub VOA of the 3A-algebra. We will show in Corollary 3.7 in a general fashion that there exists a natural action of a 3-transposition group on the commutant of the 3A-algebra. Our construction of 3-transposition groups is not covered by the method developed in [Ma05]. Although we will consider a commutant subalgebra, it is worthy to mention that our 3-transposition group is indeed a subgroup of the automorphism group of the whole VOA. As an application, we will obtain a realization of  $\text{Fi}_{23}$  as an automorphism group of the moonshine VOA. Our construction naturally reflects a subgroup structure of the Monster. Inside the Monster,  $\text{Fi}_{24}$  is not a subgroup but its extension  $3.\text{Fi}_{24}$  is a subgroup, therefore in [HLY12b] we have to use  $\sigma$ -involutions which are defined locally on the commutant subalgebra to realize  $\text{Fi}_{24}$ . In this article, we will consider a series of subalgebras called the (2A,3A)-generated subalgebras and consider their commutant subalgebras inductively. Our argument works in a general setting and is compatible with inductive structures of 3-transposition groups; thereby applying to the moonshine VOA, we will obtain a series of sub VOAs affording the actions of the Fischer 3-transposition groups. Finally, we will establish the Conway-Miyamoto correspondences for  $\text{Fi}_{23}$  and  $\text{Fi}_{22}$  using our construction. In principle, we can continue our argument to the next case  $\text{Fi}_{21} \cong \text{PSU}_6(2)$  where transpositions of  $\text{Fi}_{21}$  are described by  $c = c_7$  Virasoro vectors. However, by a technical reason, we cannot determine the Griess algebra and its invariant subalgebra of the centralizer (cf. Remark 5.16). The Griess algebra corresponding to  $\text{Fi}_{21}$  is rather small and the Conway-Miyamoto correspondence seems to terminate at  $\text{Fi}_{22}$  in this series.

The organization of this article is as follows. In Section 2, we recall some basic properties about Virasoro VOAs [FZ92, W93] and the dihedral algebras constructed in [LYY07, LYY05]. In Section 3, we give a construction of 3-transposition groups using the 3A-algebra. We first fix a pair of Ising vectors  $a, b \in V$  such that  $a$  and  $b$  generate a 3A-algebra in  $V$  and then consider the set of Ising vectors  $I_{a,b}$  of  $V$  such that  $\langle a, x \rangle$  and  $\langle b, x \rangle$  are isomorphic to the 2A-algebra for all  $x \in I_{a,b}$ . We will show that the Miyamoto

involutions associated to Ising vectors of  $I$  generate a 3-transposition group in  $\text{Aut}(V)$ . In Section 4, we will determine all possible structures of the sub VOA generated by  $a, b, x, y$  for any  $x, y \in I_{a,b}$  and then introduce the (2A,3A)-generated subalgebras  $X^{[i]}$  for  $i \geq 0$ . The commutant subalgebras of  $X^{[i]}$  are used to study inductive 3-transposition subgroups. In Section 5, we will apply our results to the moonshine VOA and Fischer 3-transposition groups  $\text{Fi}_{23}$  and  $\text{Fi}_{22}$ . The Conway-Miyamoto correspondence between the 2A-involutions of  $\text{Fi}_{23}$  and  $c = c_5$  Virasoro vectors of  $\text{Com}_{V^\natural} X^{[0]}$  and that between the 2A-involutions of  $\text{Fi}_{22}$  and  $c = c_6$  Virasoro vectors of  $\sigma$ -type of  $\text{Com}_{V^\natural} X^{[1]}$  will be established. In the appendix, we will give explicit constructions of the VOAs mentioned in Section 4.

The authors used computer algebra systems Risa/Asir for the calculations in Griess algebras and GAP 4.7.4 for Linux for the character calculations of finite groups.

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**Notation and terminology.** In this paper, vertex operator algebras (VOAs) are defined over the complex number field  $\mathbb{C}$ . A VOA  $V$  is called of *OZ-type* if it has the  $L(0)$ -grading  $V = \bigoplus_{n \geq 0} V_n$  such that  $V_0 = \mathbb{C}\mathbb{1}$  and  $V_1 = 0$ . In this case,  $V$  is equipped with a unique invariant bilinear form such that  $(\mathbb{1}|\mathbb{1}) = 1$ . A real form  $V_{\mathbb{R}}$  of  $V$  is called *compact* if the associated bilinear form is positive definite. We will mainly consider a VOA of OZ-type having a compact real form. For a subset  $A$  of  $V$ , the subalgebra generated by  $A$  is denoted by  $\langle A \rangle$ . For  $a \in V_n$  we define  $\text{wt}(a) = n$ . We write  $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$  for  $a \in V$  and define its *zero-mode* by  $\text{o}(a) := a_{(\text{wt}(a)-1)}$  if  $a$  is homogeneous and extend linearly. The weight two subspace  $V_2$  carries a structure of a commutative algebra defined by the product  $\text{o}(a)b = a_{(1)}b$  for  $a, b \in V_2$ . This algebra is called the *Griess algebra* of  $V$ . A *Virasoro vector* is  $a \in V_2$  such that  $a_{(1)}a = 2a$ . The subalgebra  $\langle a \rangle$  is isomorphic to a Virasoro VOA with central charge  $2(a|a)$ . If two Virasoro vectors  $a$  and  $b$  are orthogonal, we will denote their sum by  $a \dot{+} b$ . We denote by  $L(c, h)$  the irreducible highest weight module over the Virasoro algebra with central charge  $c$  and highest weight  $h$ . A *simple*

$c = c_e$  Virasoro vector  $e \in V$  is a Virasoro vector such that  $\langle e \rangle \cong L(c_e, 0)$ . If  $e$  is taken from a compact real form of  $V$  then  $e$  is always simple. A Virasoro vector  $\omega$  is called the *conformal vector* of  $V$  if each graded subspace  $V_n$  agrees with  $\text{Ker}_V(\text{o}(\omega) - n)$  and satisfies  $\omega_{(0)}a = a_{(-2)}\mathbb{1}$  for all  $a \in V$ . The half of the conformal vector gives the unit of the Griess algebra and hence uniquely determined. We write  $\omega_{(n+1)} = L(n)$  for  $n \in \mathbb{Z}$ . A Virasoro vector  $e$  of  $V$  is called *characteristic* if it is fixed by  $\text{Aut}(V)$ . Clearly the conformal vector is characteristic in  $V$ . A *sub VOA*  $(W, e)$  of  $V$  is a pair of a subalgebra  $W$  of  $V$  together with a Virasoro vector  $e$  in  $W$  such that  $e$  is the conformal vector of  $W$ . We often omit  $e$  and simply denote by  $W$ . The commutant subalgebra of  $(W, e)$  in  $V$  is defined by  $\text{Com}_V W := \text{Ker}_V e_{(0)}$  (cf. [FZ92]). A sub VOA  $W$  of  $V$  is called *full* if  $V$  and  $W$  shares the same conformal vector. For a subgroup  $G$  of  $\text{Aut}(V)$ , we denote the set of  $G$ -invariants by  $V^G$ .

## 2 The dihedral subalgebras

First we recall some basic properties about Virasoro VOAs [FZ92, W93] and the dihedral algebras constructed in [LYY07, LYY05].

### 2.1 Virasoro vertex operator algebras

Let

$$\begin{aligned} c_n &:= 1 - \frac{6}{(n+2)(n+3)}, \quad n = 1, 2, 3, \dots, \\ h_{r,s}^{(n)} &:= \frac{\{r(n+3) - s(n+2)\}^2 - 1}{4(n+2)(n+3)}, \quad 1 \leq r \leq n+1, \quad 1 \leq s \leq n+2. \end{aligned} \tag{2.1}$$

It is shown in [W93] that  $L(c_n, 0)$  is rational and  $L(c_n, h_{r,s}^{(n)})$ ,  $1 \leq s \leq r \leq n+1$ , are all the irreducible  $L(c_n, 0)$ -modules (see also [DMZ94]). This is the so-called unitary series of the Virasoro VOAs. Note that  $h_{r,s}^{(n)} = h_{n+2-r, n+3-s}^{(n)}$ . The fusion rules among  $L(c_n, 0)$ -modules are computed in [W93] and given by

$$L(c_n, h_{r_1, s_1}^{(n)}) \boxtimes L(c_n, h_{r_2, s_2}^{(n)}) = \sum_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} L(c_n, h_{|r_1 - r_2| + 2i - 1, |s_1 - s_2| + 2j - 1}^{(n)}), \tag{2.2}$$

where  $M = \min\{r_1, r_2, n+2-r_1, n+2-r_2\}$  and  $N = \min\{s_1, s_2, n+3-s_1, n+3-s_2\}$ .

**Definition 2.1.** A Virasoro vector  $e$  with central charge  $c$  is called *simple* if  $\langle e \rangle \cong L(c, 0)$ . A simple  $c = 1/2$  Virasoro vector is called an *Ising vector*.

The fusion rules among  $L(c_m, 0)$ -modules have a canonical  $\mathbb{Z}_2$ -symmetry which gives rise to an involutive automorphism of a VOA.



**Theorem 2.2** ([Mi96]). *Let  $V$  be a VOA and  $e \in V$  a simple Virasoro vector with central charge  $c_n$ . Denote by  $V_e[h_{r,s}^{(n)}]$  the sum of irreducible  $\langle e \rangle \cong L(c_n, 0)$ -submodules isomorphic to  $L(c_n, h_{r,s}^{(n)})$ ,  $1 \leq s \leq r \leq n+1$ . Then the linear map*

$$\tau_e := (-1)^{4(n+2)(n+3)o(e)} = \begin{cases} (-1)^{r+1} & \text{on } V_e[h_{r,s}^{(n)}] \text{ if } n \text{ is even,} \\ (-1)^{s+1} & \text{on } V_e[h_{r,s}^{(n)}] \text{ if } n \text{ is odd,} \end{cases}$$

*defines an automorphism of  $V$ .*

The automorphism  $\tau_e$  is called the *Miyamoto involution* associated to  $e$ .

**Definition 2.3.** Let  $e$  be a simple  $c = c_n$  Virasoro vector of  $V$ . Set

$$P_n := \begin{cases} \{h_{1,s}^{(n)} \mid 1 \leq s \leq n+2\} & \text{if } n \text{ is even,} \\ \{h_{r,1}^{(n)} \mid 1 \leq r \leq n+1\} & \text{if } n \text{ is odd.} \end{cases}$$

It follows from the fusion rules in (2.3) that the subspace

$$V_e[P_n] = \bigoplus_{h \in P_n} V_e[h]$$

forms a subalgebra of  $V$ . We say that  $e$  is of  $\sigma$ -type on  $V$  if  $V = V_e[P_n]$ .

**Theorem 2.4** ([Mi96]). *The linear map*

$$\sigma_e := \begin{cases} (-1)^{s+1} & \text{on } V_e[h_{1,s}^{(n)}] \text{ if } n \text{ is even,} \\ (-1)^{r+1} & \text{on } V_e[h_{r,1}^{(n)}] \text{ if } n \text{ is odd,} \end{cases}$$

*defines an element of  $\text{Aut}(V_e[P_n])$ .*

The automorphism  $\sigma_e$  is called the Miyamoto involution of  $\sigma$ -type or simply the  $\sigma$ -involution. Suppose  $e$  is an Ising vector, i.e., simple  $c = 1/2$  Virasoro vector. Then we have

$$\tau_e = \begin{cases} 1 & \text{on } V_e[0] \oplus V_e[1/2], \\ -1 & \text{on } V_e[1/16]. \end{cases} \quad (2.3)$$

If  $e$  is of  $\sigma$ -type on  $V$ , the linear map  $\sigma_e$  is defined by

$$\sigma_e := \begin{cases} 1 & \text{on } V_e[0], \\ -1 & \text{on } V_e[1/2]. \end{cases} \quad (2.4)$$

Alternatively, one can define  $\sigma_e = (-1)^{2o(e)}$ . By the definition of Miyamoto involutions we have the following conjugation.



**Proposition 2.5.** *Let  $e$  be a simple  $c = c_n$  Virasoro vector of  $V$ . For  $g \in \text{Aut}(V)$  we have  $\tau_{ge} = g\tau_e g^{-1}$ . If  $e$  is of  $\sigma$ -type on  $V$ , then  $\sigma_{ge} = g\sigma_e g^{-1}$ .*

For an Ising vector  $e$  and  $v \in V_2$ , one has  $v + \tau_e v \in V^{\langle \tau_e \rangle}$  and hence  $\sigma_e(v + \tau_e v)$  is a well-defined element in  $V_2$ . Then the following holds (cf. Eq. (2.2) of [S07]):

$$e_{(1)}v = 8(e \mid v)e + \frac{5}{32}v + \frac{3}{32}\tau_e v - \frac{1}{8}\sigma_e(v + \tau_e v). \quad (2.5)$$

If  $\tau_e v = v$  then we also have the following.

$$\sigma_e v = v + 32(e \mid v)e - 4e_{(1)}v. \quad (2.6)$$

These relations will be used in Section 3.

In [LYY07, LYY05] subalgebras generated by two Ising vectors are constructed using the  $E_8$ -lattice and such subalgebras are classified in [S07] at the level of Griess algebras.

**Theorem 2.6** ([S07]). *Let  $V$  be a VOA of OZ-type with compact real form  $V_{\mathbb{R}}$  and let  $e$  and  $f$  be distinct Ising vectors of  $V_{\mathbb{R}}$ . Then the Griess algebra of the subalgebra  $\langle e, f \rangle$  is isomorphic to one of the eight algebras called 2A, 3A, 4A, 5A, 6A, 4B, 2B and 3C-type constructed in [LYY07, LYY05]. The inner product and the dimension of the Griess algebra are as follows<sup>1</sup>.*

Type	1A	2A	3A	4A	5A	6A	4B	2B	3C
$2^{10}(e \mid f)$	$2^8$	$2^5$	13	8	6	5	4	0	4
$\dim \langle e, f \rangle_2$	1	3	4	5	6	8	5	2	3
# of Ising vectors	1	3	3	4	5	7	4	2	3

We call  $\langle e, f \rangle$  the *dihedral subalgebra* and we denote the dihedral subalgebra of type  $nX$  by  $U_{nX}$ . As a by-product of the classification of dihedral subalgebras, the following 6-transposition property was established in [S07].

**Theorem 2.7** ([S07]). *Let  $V$  be a VOA of OZ-type with compact real form  $V_{\mathbb{R}}$  and let  $e$  and  $f$  be distinct Ising vectors of  $V_{\mathbb{R}}$ . Then*

- (1) *If  $\langle e, f \rangle$  is 2A or 2B-type then  $|\tau_e \tau_f|$  divides 2.*
- (2) *If  $\langle e, f \rangle$  is 3A or 3C-type then  $|\tau_e \tau_f| = 3$ .*
- (3) *If  $\langle e, f \rangle$  is 4A or 4B-type then  $|\tau_e \tau_f| = 2$  or 4.*
- (4) *If  $\langle e, f \rangle$  is 5A-type then  $|\tau_e \tau_f| = 5$ .*
- (5) *If  $\langle e, f \rangle$  is 6A-type then  $|\tau_e \tau_f| = 3$  or 6.*
- (6)  *$\tau_e f = f$  if and only if  $\langle e, f \rangle$  is either 2A or 2B-type.*
- (7)  *$\tau_e f = \tau_f e$  if and only if  $\langle e, f \rangle$  is either 3A or 3C-type.*

*In particular, the order of  $\tau_e \tau_f$  is bounded by 6.*

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<sup>1</sup>Here we include the case  $e = f$  which is called the 1A-type.

The next lemma also follows from the classification of the dihedral algebras.

**Lemma 2.8.** *Let  $a^1, a^2, b^1, b^2$  be Ising vectors and set  $U^1 = \langle a^1, b^1 \rangle$  and  $U^2 = \langle a^2, b^2 \rangle$ . If  $U^1$  is a proper subalgebra of  $U^2$  then the types of  $(U^1, U^2)$  are either (2A, 4B), (2A, 6A), (2B, 4A) or (3A, 6A).*

In the rest of this section, we will summarize some properties of the dihedral subalgebras of 2A, 3A and 6A-types which will be used in the later sections. See [LYY05] for details.

## 2.2 2A-algebra

It follows from Theorem 2.6 that  $\langle a, b \rangle$  is of 2A-type if and only if  $(a | b) = 2^{-5}$ . Let  $U_{2A} = \langle a, b \rangle$  be the 2A-algebra. Then  $a$  and  $b$  are of  $\sigma$ -type on  $\langle a, b \rangle$  and satisfies  $\sigma_a b = \sigma_b a$ . Set  $a \circ b := \sigma_a b = \sigma_b a$ . It follows from (2.6) that

$$a \circ b = a + b - 4a_{(1)}b. \quad (2.7)$$

**Theorem 2.9.** *Let  $U_{2A} = \langle a, b \rangle$  be the 2A-algebra.*

- (1) *There are exactly three Ising vectors in of  $U_{2A}$ , namely,  $a, b$  and  $a \circ b$ .*
- (2) *The Griess algebra of  $U_{2A}$  is 3-dimensional spanned by these three Ising vectors.*
- (3)  *$(a | b) = (a | a \circ b) = (b | a \circ b) = 2^{-5}$ .*
- (4)  *$\text{Aut}(U_{2A}) = \langle \sigma_a, \sigma_b \rangle \cong S_3$ .*
- (5) *Suppose  $U_{2A} = \langle a, b \rangle$  is a subalgebra of a VOA  $V$ . Then  $\tau_{a \circ b} = \tau_a \tau_b = \tau_b \tau_a$  on  $V$ .*

We will call the set of Ising vectors  $\{a, b, a \circ b\}$  of  $\langle a, b \rangle$  a *2A-triple*. The subalgebra generated by three Ising vectors of  $\sigma$ -type was classified in [Ma05] and the following “No 2A-tetrahedron lemma” holds (cf. Proposition 1 of [Ma05]<sup>2</sup>).

**Lemma 2.10.** *Let  $a, b, c$  be Ising vectors such that  $(a | b) = (a | c) = (b | c) = 2^{-5}$  and  $c \notin \langle a, b \rangle$ . Then we have  $(a \circ b | c) = (a | b \circ c) = (b | a \circ c) = 0$ .*

## 2.3 3A-algebra

It follows from Theorem 2.6 that  $\langle a, b \rangle$  is of 3A-type if and only if  $(a | b) = 13 \cdot 2^{-10}$ . Let  $U_{3A} = \langle a, b \rangle$  be the 3A-algebra. Then  $\tau_a b = \tau_b a$  holds. Set  $c = \tau_a b$  and define

$$u = u_{a,b} = \frac{2^6}{135} (2a + 2b + c - 16a_{(1)}b). \quad (2.8)$$

Then  $u$  is a  $c = 4/5$  Virasoro vector in  $U_{3A}$ .

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<sup>2</sup>Proposition 3.3.8 of [arXiv:math/0311400](https://arxiv.org/abs/math/0311400).

**Theorem 2.11.** *Let  $U_{3A} = \langle a, b \rangle$  be the 3A-algebra.*

- (1) *There are exactly three Ising vectors in  $U_{3A}$ , namely,  $a, b$  and  $c$ .*
- (2)  *$\text{Aut}(U_{3A}) = \langle \tau_a, \tau_b \rangle \cong S_3$ .*
- (3) *The Griess algebra of  $U_{3A}$  is 4-dimensional and spanned by  $u, a, b$  and  $c$ .*
- (4) *The  $c = 4/5$  Virasoro vector  $u$  is characteristic in  $U_{3A}$ .*
- (5) *The multiplications and inner products of the Griess algebra are as follows.*

$$\begin{aligned} a_{(1)}b &= -\frac{135}{2^{10}}u + \frac{1}{2^3}a + \frac{1}{2^3}b + \frac{1}{2^4}c, & a_{(1)}u &= \frac{5}{2^4}u + \frac{4}{3^2}a - \frac{2}{3^2}b - \frac{2}{3^2}c, \\ (a|b) &= (a|c) = (b|c) = \frac{13}{2^{10}}, & (a|u) &= (b|u) = (c|u) = \frac{1}{2^4}. \end{aligned}$$

- (6) *There is a unique characteristic  $c = 6/7$  Virasoro vector in  $\langle a, b \rangle$ , which is given by*

$$v = -\frac{5}{14}u + \frac{16}{21}(a + b + c).$$

Moreover,  $u$  and  $v$  are mutually orthogonal and  $u + v$  is the conformal vector of  $\langle a, b \rangle$ .

We call the set of Ising vectors in  $U_{3A}$  a 3A-triple. We will need the following relations later.

**Lemma 2.12.** *In the 3A-algebra, the following holds.*

$$\sigma_a u = -\frac{1}{2^2}u + \frac{2}{3^2}a + \frac{8}{3^2}(b + c), \quad \sigma_a(b + c) = \frac{135}{2^7}u - \frac{3}{2^4}a + \frac{1}{2^2}(b + c).$$

## 2.4 6A-algebra

It follows from Theorem 2.6 that  $\langle a, b \rangle$  is of 6A-type if and only if  $(a|b) = 5 \cdot 2^{-10}$ . Let  $U_{6A} = \langle a, b \rangle$  be the 6A-algebra. There are 7 Ising vectors in the 6A-algebra. Namely,  $a, b, \tau_a b, \tau_b a, \tau_a \tau_b a, \tau_b \tau_a b$  and

$$x = a \circ \tau_b \tau_a b = b \circ \tau_a \tau_b a = \tau_b a \circ \tau_a b. \quad (2.9)$$

The Ising vector  $x$  is of  $\sigma$ -type and defines a  $\sigma$ -involution on  $\langle a, b \rangle$ . Inside the 6A-algebra, there are two 3A-subalgebras and the 3-sets  $\{a, \tau_b a, \tau_a \tau_b a\}$  and  $\{b, \tau_a b, \tau_b \tau_a b\}$  form the 3A-triples with the same characteristic  $c = 4/5$  Virasoro vector  $u_{a, \tau_b a} = u_{b, \tau_a b}$ . The automorphism  $\tau_a \tau_b$  acts as a 3-cycle on each of the 3A-triples and has order 3 on  $\langle a, b \rangle$ .

**Theorem 2.13.** *Let  $U_{6A} = \langle a, b \rangle$  be the 6A-algebra.*

- (1) *The set of Ising vectors of  $U_{6A}$  is given by  $\{x, a, b, \tau_a b, \tau_b a, \tau_a \tau_b a, \tau_b \tau_a b\}$ .*
- (2)  *$x$  is the unique characteristic  $\sigma$ -type Ising vector in  $U_{6A}$  and  $\sigma_x$  is central in  $\text{Aut}(U_{6A})$ .*
- (3)  *$\{x, a, \tau_b \tau_a b\}, \{x, b, \tau_a \tau_b a\}$  and  $\{x, \tau_b a, \tau_a b\}$  form 2A-triples in  $\langle a, b \rangle$ .*

(4)  $\langle a, \tau_b a, \tau_a \tau_b a \rangle$  and  $\langle b, \tau_a b, \tau_b \tau_a b \rangle$  are isomorphic to the 3A-algebra with the same characteristic  $c = 4/5$  Virasoro vector  $u = u_{a, \tau_b a} = u_{b, \tau_a b}$ .

(5) The Griess algebra of  $U_{6A}$  is 8-dimensional with a basis  $\{u, x, a, b, \tau_a b, \tau_b a, \tau_a \tau_b a, \tau_b \tau_a b\}$ .

(6)  $\text{Aut}(U_{6A}) = \langle \tau_a \sigma_x, \tau_b \rangle$  is isomorphic to the dihedral group of order 12.

(7) Set  $\rho = \tau_a \tau_b \sigma_x$  and  $e^i = \rho^i a$ . Then  $\{a, b, \tau_a b, \tau_b a, \tau_a \tau_b a, \tau_b \tau_a b\} = \{e^i \mid 0 \leq i \leq 5\}$  and the multiplications and inner products of the Griess algebra are as follows.

$$\begin{aligned} u_{(1)}x &= 0, & u_{(1)}e^i &= -\frac{135}{2^{10}}u + \frac{4}{3^2}e^i - \frac{2}{3^2}(e^{i+2} + e^{i+4}), & (u \mid x) &= 0, & (u \mid e^i) &= \frac{1}{2^4}, \\ x_{(1)}e^i &= \frac{1}{4}x + e^i - e^{i+3}, & e_{(1)}^0e^1 &= \frac{45}{2^{10}}u + \frac{1}{2^5}(x + e^0 + e^1 - e^2 - e^3 - e^4 - e^5), \\ (x \mid e^i) &= \frac{1}{2^5}, & (e^i \mid e^{i+1}) &= \frac{5}{2^{10}}, & (e^i \mid e^{i+2}) &= \frac{13}{2^{10}}, & (e^i \mid e^{i+3}) &= \frac{1}{2^5}. \end{aligned}$$

(8) The commutant of  $\langle a, \tau_b a \rangle$  in  $\langle a, b \rangle$  is generated by a  $c = 25/28$  Virasoro vector

$$f = -\frac{15}{56}u + \frac{1}{2}x - \frac{2}{21}(a + \tau_b a + \tau_a \tau_b a) + \frac{2}{3}(b + \tau_a b + \tau_b \tau_a b).$$

Therefore,  $\langle a, b \rangle$  has a unitary Virasoro frame isomorphic to  $L(4/5, 0) \otimes L(6/7, 0) \otimes L(25/28, 0)$ .

The characteristic Ising vector  $x$  is called *central* since  $\sigma_x$  is central in the automorphism group of the 6A-algebra. We collect some technical relations needed in the later arguments.

**Lemma 2.14.** *Let  $\langle a, b \rangle$  be the 6A-algebra and let  $u$  be the characteristic  $c = 4/5$  Virasoro vector and  $x$  the central Ising vector. Then one has*

$$\begin{aligned} a_{(1)}(\tau_b a)_{(1)}x &= -\frac{45}{2^{10}}u + \frac{1}{2^7}(7x + 11a + b + 5\tau_b a - 7\tau_b \tau_a b + 3\tau_a \tau_b a - \tau_a b), \\ \sigma_a(b + \tau_a b) &= -\frac{45}{2^7}u_{a,b} - \frac{1}{2^2}x + \frac{1}{2^4}a + \frac{1}{2^2}\tau_b \tau_a b + b + \tau_a b + \frac{1}{2^2}(\tau_b a + \tau_a \tau_b a). \end{aligned}$$

**Remark 2.15.** In some situation, it is useful to display the relations above using another labeling. We rename the Ising vectors so that  $a$  and  $b$  generate a 3A-algebra and  $x$  is the central Ising vector in a 6A-algebra. Let  $c = \tau_a b = \tau_b a$ . Then  $\{a, b, c\}$  forms a 3A-triple and  $\{a \circ x, b \circ x, c \circ x\}$  gives another 3A-triple. Lemma 2.14 can be rewritten as follows.

$$\begin{aligned} a_{(1)}b_{(1)}x &= -\frac{45}{2^{10}}u_{a,b} + \frac{1}{2^7}(7x + 11a + 5b + 3c - 7a \circ x - b \circ x + c \circ x), \\ \sigma_a(b \circ x + c \circ x) &= -\frac{45}{2^7}u - \frac{1}{2^2}x + \frac{1}{2^4}a + \frac{1}{2^2}a \circ x + \frac{1}{2^2}(b + c) + b \circ x + c \circ x. \end{aligned} \tag{2.10}$$

### 3 3-transposition property

Suppose that  $V$  has a grading  $V = \bigoplus_{n \geq 0} V_n$  with  $V_0 = \mathbb{C}1$  and  $V_1 = 0$ . We also assume that  $V$  has a compact real form  $V_{\mathbb{R}}$  and all Ising vectors are taken from it.

**Lemma 3.1.** *Let  $e$  be an Ising vector and  $v \in V_2$ . Then  $\tau_e v \in \langle e, v \rangle$ . If  $v \in V^{\langle \tau_e \rangle}$ , then  $\sigma_e v \in \langle e, v \rangle$ .*

**Proof:** We have the eigenspace decomposition

$$V_2 = \mathbb{C}e \oplus (V_2)_e[0] \oplus (V_2)_e[1/2] \oplus (V_2)_e[1/16].$$

Decompose  $v = \lambda e + v_0 + v_{1/2} + v_{1/16}$  with  $\lambda \in \mathbb{C}$  and  $v_h \in (V_2)_e[h]$ . Then we have

$$16e_{(1)}v = 32\lambda e + 8v_{1/2} + v_{1/16} \quad \text{and} \quad 256e_{(1)}e_{(1)}v = 1024\lambda e + 64v_{1/2} + v_{1/16}.$$

It follows that all of  $v_0, v_{1/2}, v_{1/16}$  are in  $\text{Span}\{v, e, e_{(1)}v, e_{(1)}e_{(1)}v\}$ . Therefore,  $\tau_e v = v - 2v_{1/16} \in \langle e, v \rangle$ . If  $v \in V^{\langle \tau_e \rangle}$  then it follows from (2.6) that  $\sigma_e v \in \langle e, v \rangle$ . ■

**Lemma 3.2.** *Let  $a, b, x$  be Ising vectors of  $V_{\mathbb{R}}$  such that  $\tau_a b = \tau_b a$  and  $\langle a, x \rangle \cong \langle b, x \rangle \cong U_{2A}$ . Then  $\langle a, b \rangle \cong U_{3A}$ ,  $\langle a, b, x \rangle \cong U_{6A}$  and  $x$  is the central Ising vector of  $\langle a, b, x \rangle$ .*

**Proof:** Since  $a, b \in V^{\langle \tau_x \rangle}$ ,  $x$  is of  $\sigma$ -type on  $\langle a, b, x \rangle$  and one has  $\tau_{a \circ x} = \tau_a \tau_x$ . We will show that  $\langle a \circ x, b \rangle = \langle a, b, x \rangle$ . By Lemma 3.1,  $\langle a \circ x, b \rangle \subset \langle a, b, x \rangle$ . Since  $\tau_x b = b$ , we have  $\tau_{a \circ x} b = \tau_a \tau_x b = \tau_a b$ . By our assumption, we have  $\tau_a b = \tau_b a$ . Then  $\tau_{a \circ x} b = \tau_a \tau_x b = \tau_a b = \tau_b a$  and  $a = \tau_b \tau_{a \circ x} b \in \langle a \circ x, b \rangle$  by Lemma 3.1. Since  $a \circ (a \circ x) = x$ , we also have  $x = a + a \circ x - 4a_{(1)}(a \circ x) \in \langle a \circ x, b \rangle$ . Therefore,  $a, b, x \in \langle a \circ x, b \rangle$  and  $\langle a, b, x \rangle = \langle a \circ x, b \rangle$ . By (7) of Theorem 2.7, the assumption  $\tau_a b = \tau_b a$  implies  $\langle a, b \rangle$  is isomorphic to either  $U_{3A}$  or  $U_{3C}$ . Since  $\langle a, b \rangle \subset \langle a \circ x, b \rangle$ , the possible types of  $\langle a, b \rangle$  and  $\langle a \circ x, b \rangle$  are only  $U_{3A}$  and  $U_{6A}$ , respectively, by Lemma 2.8. As  $x$  is of  $\sigma$ -type on  $\langle a, b, x \rangle \cong U_{6A}$ ,  $x$  is the unique central Ising vector in  $\langle a, b, x \rangle$ . This completes the proof. ■

**Lemma 3.3.** *Let  $a, b, x$  be Ising vectors of  $V_{\mathbb{R}}$  and such that  $\langle a, x \rangle \cong \langle b, x \rangle \cong U_{2A}$  and  $\langle a, b \rangle \cong U_{6A}$ . Let  $z$  be the central Ising vector of  $\langle a, b \rangle$ . Then  $\tau_x = \tau_z$  on  $V_{\mathbb{R}}$ .*

**Proof:** Suppose  $\tau_x \neq \tau_z$  on  $V_{\mathbb{R}}$ . Let  $a' = \tau_b \tau_a b$ . Then  $a, a'$  and  $z = a \circ a'$  form a 2A-triple by (3) of Theorem 2.13. By Proposition 2.5 and (5) of Theorem 2.9, we have

$$\tau_z = \tau_a \tau_{a'} = \tau_a \tau_b \tau_a b = (\tau_a \tau_b)^3.$$

Since  $\langle a, x \rangle$  and  $\langle b, x \rangle$  are of 2A-type, we have  $[\tau_x, \tau_a] = [\tau_x, \tau_b] = 1$ . Then  $\tau_x$  commutes with  $\tau_z = (\tau_a \tau_b)^3$  and  $\tau_{a'} = \tau_a \tau_z$ . Therefore,  $G = \langle \tau_a, \tau_{a'}, \tau_x \rangle \subset \text{Aut}(V_{\mathbb{R}})$  is an abelian subgroup. The order of  $\tau_a \tau_b$  is either 3 or 6 on  $V_{\mathbb{R}}$ . Since

$$(\tau_{a \circ x} \tau_b)^3 = (\tau_x \tau_a \tau_b)^3 = \tau_x^3 (\tau_a \tau_b)^3 = \tau_x \tau_z \neq 1,$$

the order of  $\tau_{a \circ x} \tau_b$  is 6 by the 6-transposition property and we have  $\langle a \circ x, b \rangle \cong U_{6A}$  by Theorem 2.7. It follows from (3) of Theorem 2.13 that  $a \circ x$  and  $\tau_b \tau_{a \circ x} b$  generates a 2A-subalgebra in  $\langle a \circ x, b \rangle$ . On the other hand, by (5) of Theorem 2.9 we have  $\tau_b \tau_{a \circ x} b = \tau_b \tau_a \tau_x b = \tau_b \tau_a b = a'$ . Therefore  $(a \circ x | a') = (a \circ x | \tau_b \tau_{a \circ x} b) = 2^{-5}$ . Since  $a$  and  $b$  are fixed by  $\tau_x$ ,  $z = a \circ a'$  is also fixed by  $\tau_x$ . Clearly  $a$  and  $a'$  are fixed by  $\langle \tau_a, \tau_{a'} \rangle$  and hence  $a, a', z \in V_{\mathbb{R}}^G$ . Similarly,  $x$  is fixed by  $\langle \tau_a, \tau_b \rangle$  so that  $x$  is fixed by  $\tau_a$  and  $\tau_{a'} = \tau_{\tau_b \tau_a b} = \tau_b \tau_a \tau_b \tau_a \tau_b$ . Therefore, all of  $a, a', z$  and  $x$  belong to  $V_{\mathbb{R}}^G$  and of  $\sigma$ -type on it. We have  $(a' | x) = (\tau_b \tau_a b | x) = (b | \tau_a \tau_b x) = (b | x) = 2^{-5}$  and  $(z | x) = (a \circ a' | x) = (\sigma_{a'} a | x) = (a | \sigma_{a'} x) = (a | \sigma_x a') = (\sigma_x a | a') = (a \circ x | a') = 2^{-5}$ . Therefore, we have  $(a | a') = (a | z) = (a | x) = (a' | z) = (a' | x) = (z | x) = 2^{-5}$ . Namely,  $a, a', x$  and  $z$  form a 2A-tetrahedron in  $V_{\mathbb{R}}^G$  which contradicts Lemma 2.10. Thus  $\tau_x = \tau_z$ . ■

**Lemma 3.4.** *Let  $a, b, x, y$  be Ising vectors of  $V$  such that  $\langle a, x \rangle \cong \langle b, x \rangle \cong \langle a, y \rangle \cong \langle b, y \rangle \cong U_{2A}$  and  $\langle x, y \rangle \cong U_{6A}$ . Then  $\tau_a = \tau_b$ .*

**Proof:** Let  $z$  be the central Ising vector of  $\langle x, y \rangle$ . Applying Lemma 3.3 to  $x, y, a$ , we obtain  $\tau_a = \tau_z$ , and applying Lemma 3.3 to  $x, y, b$ , we obtain  $\tau_b = \tau_z$ . Thus  $\tau_a = \tau_b$ . ■

Let  $E$  be the set of Ising vectors of  $V_{\mathbb{R}}$ . Suppose there is a pair  $a, b \in E$  such that  $\langle a, b \rangle \cong U_{3A}$ . Then we set

$$I_{a,b} := \{x \in E \mid \langle a, x \rangle \cong \langle b, x \rangle \cong U_{2A}\}. \quad (3.1)$$

We will show that a 3-transposition group arises from  $I_{a,b}$ .

**Theorem 3.5.** *Let  $a, b \in V_{\mathbb{R}}$  be Ising vectors such that  $\langle a, b \rangle \cong U_{3A}$ . Define  $I_{a,b}$  as in (3.1). Then for any  $x, y \in I_{a,b}$ ,  $\langle x, y \rangle \cong U_{1A}, U_{2A}, U_{2B}$  or  $U_{3A}$ . Moreover, if  $\langle x, y \rangle \cong U_{2B}$ , then  $\langle a \circ x, y \rangle \cong U_{2A}$ .*

**Proof:** Let  $x, y \in I_{a,b}$ . We eliminate the possibility of the type of  $\langle x, y \rangle$  other than 1A, 2A, 2B and 3A. By our choice,  $\tau_x$  and  $\tau_y$  are commutative with  $\tau_a$  and  $\tau_b$ .

(i) Case 6A: Since  $\tau_a \tau_b$  is non-trivial on  $\langle a, b \rangle \cong U_{3A}$ , it is also non-trivial on  $V$ . If  $\langle x, y \rangle \cong U_{6A}$ , then  $\tau_a = \tau_b$  by Lemma 3.4 and we have a contradiction.

(ii) Case 5A: Suppose  $\langle x, y \rangle \cong U_{5A}$ . Then  $\tau_x \tau_y$  has an order 5. Since  $\tau_a$  is non-trivial,  $\tau_a \tau_x \tau_y = \tau_{a \circ x} \tau_y$  has order 10 and we have a contradiction by the 6-transposition property in Theorem 2.7.

(iii) Cases 4A and 4B: Suppose  $\langle x, y \rangle \cong U_{4A}$  or  $U_{4B}$ . Then  $\tau_x \tau_y$  has order 2 or 4 by Theorem 2.7. By (5) of Theorem 2.9, we have

$$\tau_{a \circ x} \tau_{b \circ y} = \tau_a \tau_x \tau_b \tau_y = \tau_a \tau_b \tau_x \tau_y.$$

If  $\tau_x \tau_y$  has order 4, then  $\tau_{a \circ x} \tau_{b \circ y}$  has order 12, which contradicts the 6-transposition property in Theorem 2.7. If  $\tau_x \tau_y$  has order 2, then  $\tau_{a \circ x} \tau_{b \circ y}$  has order 6 and the sub VOA generated by  $a \circ x$  and  $b \circ y$  is isomorphic to the 6A-algebra. Since  $\tau_{a \circ x} \tau_{b \circ y}$  has order 3 on  $\langle a \circ x, b \circ y \rangle$ ,  $(\tau_{a \circ x} \tau_{b \circ y})^3 = \tau_x \tau_y$  acts trivially on  $\langle a \circ x, b \circ y \rangle$ . Thus  $\tau_x \tau_y$  fixes both  $a \circ x$  and  $b \circ y$ . Since  $\tau_y$  fixes  $b \circ y$ , so does  $\tau_x$  and hence  $\tau_{b \circ y}$  fixes  $x$ . Then  $\tau_y$  fixes  $x$  since  $\tau_{b \circ y} = \tau_y \tau_b$  and  $\tau_b$  fixes  $x$ . This contradicts (6) of Theorem 2.7. Hence,  $\langle x, y \rangle$  cannot be the 4A nor the 4B-algebra.

(iv) Case 3C: That  $\langle x, y \rangle \cong U_{3C}$  is impossible by Lemma 3.2 since  $\langle x, y \rangle \subset \langle a, x, y \rangle \cong U_{6A}$ .

Thus,  $\langle x, y \rangle$  is isomorphic to one of  $U_{1A}$ ,  $U_{2A}$ ,  $U_{2B}$  or  $U_{3A}$ . Finally, suppose  $\langle x, y \rangle \cong U_{2B}$ . Then  $x_{(1)}y = 0$  and  $(x|y) = 0$ . It follows from  $a \circ x = a + x - 4a_{(1)}x$  that

$$(a \circ x|y) = (a|y) + (x|y) - 4(a_{(1)}x|y) = (a|y) - 4(a|x_{(1)}y) = 2^{-5} - 4(a|0) = 2^{-5}$$

and hence  $\langle a \circ x, y \rangle$  is isomorphic to  $U_{2A}$ . This completes the proof.  $\blacksquare$

**Remark 3.6.** Note that  $\langle a, b, x, y \rangle = \langle a, b, a \circ x, y \rangle$ . Therefore, when we consider the the subalgebra  $\langle a, b, x, y \rangle$ , we obtain the same subalgebra in the 2A-case and in the 2B-case in Theorem 3.5.

As a corollary, we have

**Corollary 3.7.** *For  $x, y \in I_{a,b}$ , one has  $|\tau_x \tau_y| \leq 3$ . Therefore,  $G = \langle \tau_x \mid x \in I_{a,b} \rangle$  forms a 3-transposition subgroup in the stabilizer  $\{g \in \text{Aut}(V_{\mathbb{R}}) \mid ga = a, gb = b\}$  of  $\langle a, b \rangle$ .*

## 4 The subalgebra $\langle a, b, x, y \rangle$

Let  $V$  be an OZ-type VOA with a compact real form  $V_{\mathbb{R}}$ , and let  $E$  be the set of Ising vectors in  $V_{\mathbb{R}}$ . Suppose we have a pair  $a, b \in E$  such that  $(a|b) = 13 \cdot 2^{-10}$  and consider the subset  $I_{a,b}$  defined as in (3.1). Let  $x$  and  $y$  be distinct Ising vectors in  $I_{a,b}$ . In this section, we will study the subalgebra  $\langle a, b, x, y \rangle$ .

**Lemma 4.1.** *Let  $a, x, y$  be Ising vectors such that  $(a|x) = (a|y) = 2^{-5}$ .*

(1) *If  $\langle x, y \rangle$  is the 2A-algebra then  $(a|x_{(1)}y) = 2^{-6}$ .*

(2) *If  $\langle x, y \rangle$  is the 3A-algebra then  $(a|x_{(1)}y) = 5 \cdot 2^{-9}$ .*

**Proof:** (1): Suppose  $\langle x, y \rangle$  is the 2A-algebra. Then  $x_{(1)}y = \frac{1}{4}(x + y - x \circ y)$ . By Lemma 2.10, one has  $(a|x \circ y) = 0$ . Therefore

$$(a|x_{(1)}y) = \frac{1}{4}(a|x + y - x \circ y) = \frac{1}{4} \cdot 2 \cdot 2^{-5} = 2^{-6}.$$



(2): Suppose  $\langle x, y \rangle$  is the 3A-algebra. Then  $\langle a, x, y \rangle$  is the 6A-algebra with the central Ising vector  $a$  by Lemma 3.2. By (5) of Theorem 2.11, we have

$$(a | x_{(1)}y) = -\frac{135}{2^{10}}(a | u_{x,y}) + \frac{1}{2^4}(a | 2x + 2y + \tau_x y) = \frac{1}{2^4}(2 + 2 + 1) \cdot 2^{-5} = 5 \cdot 2^{-9}.$$

Thus we have the lemma. ■

**Lemma 4.2.** *Let  $a, b, x, y$  be Ising vectors such that  $(a | b) = 13 \cdot 2^{-10}$  and  $(a | x) = (a | y) = (b | x) = (b | y) = 2^{-5}$ . Then*

$$(x | a_{(1)}b_{(1)}y) = 2^{-7} (7(x | y) - 7(x | a \circ y) - (x | b \circ y) + (x | \tau_a b \circ y) + 19 \cdot 2^{-5}).$$

**Proof:** It follows from Lemma 3.2 that  $\langle a, b, y \rangle$  is the 6A-algebra with the central Ising vector  $y$ . The equality then follows from Lemma 2.14. (See also Eq. (2.10).) ■

**Proposition 4.3.** *Let  $a, b, x, y$  be Ising vectors such that  $\langle a, b \rangle$  is the 3A-algebra and  $(a | x) = (a | y) = (b | x) = (b | y) = 2^{-5}$ .*

(1) *If  $\langle x, y \rangle$  is the 2A-algebra then  $(a \circ x | b \circ y) = 5 \cdot 2^{-10}$ .*

(2) *If  $\langle x, y \rangle$  is the 3A-algebra then  $(a \circ x | b \circ y) = 13 \cdot 2^{-10}$ .*

**Proof:** By definition, we have  $a \circ x = a + x - 4a_{(1)}x$  and  $b \circ y = b + y - 4b_{(1)}y$ . So by Lemma 4.1, one has

$$\begin{aligned} (a \circ x | b \circ y) &= (a + x - 4a_{(1)}x | b + y - 4b_{(1)}y) \\ &= \underbrace{(a | b)}_{=13 \cdot 2^{-10}} + \underbrace{(a | y)}_{=2^{-5}} - 4 \underbrace{(a_{(1)}b | y)}_{=5 \cdot 2^{-9}} + \underbrace{(x | b)}_{=2^{-5}} + (x | y) - 4(x_{(1)}y | b) \\ &\quad - 4 \underbrace{(x | a_{(1)}b)}_{=5 \cdot 2^{-9}} - 4(a | x_{(1)}y) + 16(x | a_{(1)}b_{(1)}y) \\ &= (x | y) - 4(a | x_{(1)}y) - 4(b | x_{(1)}y) + 16(x | a_{(1)}b_{(1)}y) - 3 \cdot 2^{-10}. \end{aligned}$$

(1): If  $\langle x, y \rangle$  is the 2A-algebra, then  $(x | y) = 2^{-5}$  and  $(a | x_{(1)}y) = (b | x_{(1)}y) = 2^{-6}$  by Lemma 4.1. Therefore

$$(a \circ x | b \circ y) = 16(x | a_{(1)}b_{(1)}y) - 99 \cdot 2^{-10}.$$

In this case,  $(x | a \circ y) = 0$  by Lemma 2.10 and  $\tau_a$  acts trivially on  $\langle x, y \rangle$ . Thus  $\tau_a(b \circ y) = \tau_a b \circ \tau_a y = \tau_a b \circ y$  and  $(x | \tau_a b \circ y) = (x | \tau_a(b \circ y)) = (\tau_a x | b \circ y) = (x | b \circ y)$ . Then by Lemma 4.2, one has

$$(x | a_{(1)}b_{(1)}y) = 2^{-7} (7 \cdot 2^{-5} + 19 \cdot 2^{-5}) = 13 \cdot 2^{-11}.$$

Therefore,

$$(a \circ x | b \circ y) = 16 \cdot 13 \cdot 2^{-11} - 99 \cdot 2^{-10} = 5 \cdot 2^{-10}.$$

(2): If  $\langle x, y \rangle$  is the 3A-algebra then  $(x | y) = 13 \cdot 2^{-10}$  and  $(a | x_{(1)}y) = (b | x_{(1)}y) = 5 \cdot 2^{-9}$  by Lemma 4.1. Therefore

$$(a \circ x | b \circ y) = 16(x | a_{(1)}b_{(1)}y) - 35 \cdot 2^{-9}.$$

In this case,  $\langle x, a \circ y \rangle$ ,  $\langle x, b \circ y \rangle$  and  $\langle x, \tau_a b \circ y \rangle$  are all isomorphic to the 6A-algebra by Lemma 3.2. Thus  $(x | a \circ y) = (x | b \circ y) = (x | \tau_a b \circ y) = 5 \cdot 2^{-10}$ . Then by Lemma 4.2, one has

$$(x | a_{(1)}b_{(1)}y) = 2^{-7} (7 \cdot 13 \cdot 2^{-10} - 7 \cdot 5 \cdot 2^{-10} + 19 \cdot 2^{-5}) = 83 \cdot 2^{-14}.$$

Therefore,

$$(a \circ x | b \circ y) = 16 \cdot 83 \cdot 2^{-14} - 35 \cdot 2^{-9} = 13 \cdot 2^{-10}.$$

This completes the proof. ■

The following lemma is a slight modification of Lemma 3 of [Ma05] and is useful in determining the conformal vector of a sub VOA.

**Lemma 4.4.** *Let  $V_{\mathbb{R}}$  be a compact VOA of OZ-type and let  $A$  be a set of Virasoro vectors of  $V_{\mathbb{R}}$  such that the linear span of  $A$  forms a subalgebra of the Griess algebra. Then the subalgebra  $\langle A \rangle$  generated by  $A$  has a conformal vector and forms a sub VOA of  $V_{\mathbb{R}}$ . The conformal vector  $\eta$  of  $\langle A \rangle$  belongs to the linear span of  $A$  and uniquely determined by the condition  $(\eta | a) = (a | a)$  for all  $a \in A$ .*

**Proof:** First, we prove that  $\langle A \rangle$  has the conformal vector inside  $\mathbb{R}A$ , the linear span of  $A$ . Let  $\omega$  be the conformal vector of  $V_{\mathbb{R}}$  and let  $B$  be the orthogonal complement of  $\mathbb{R}A$  in  $(V_{\mathbb{R}})_2$ . Let  $\omega = \eta + \xi$  be the orthogonal decomposition with respect to  $(V_{\mathbb{R}})_2 = \mathbb{R}A \perp B$ . For any  $a \in A$ , one has

$$(a_{(1)}B | \mathbb{R}A) = (B | a_{(1)}\mathbb{R}A) \subset (B | \mathbb{R}A) = 0.$$

Therefore  $(\mathbb{R}A)_{(1)}B \subset B$ . Then

$$2\eta = \omega_{(1)}\eta = (\eta + \xi)_{(1)}\eta = \eta_{(1)}\eta + \eta_{(1)}\xi$$

and we obtain  $\eta_{(1)}\xi = 2\eta - \eta_{(1)}\eta \in B \cap \mathbb{R}A = 0$ . Thus  $\eta_{(1)}\eta = 2\eta$  and  $\eta_{(1)}\xi = 0$ . This implies  $\xi_{(1)}\xi = (\omega - \eta)_{(1)}\xi = 2\xi$  and hence  $\eta$  and  $\xi$  are Virasoro vectors of  $V_{\mathbb{R}}$  by Lemma 5.1 of [Mi96]. Since a Virasoro vector  $\eta$  satisfies  $\eta_{(0)}\eta = \eta_{(-2)}\mathbb{1}$ , we have

$$\xi_{(0)}\eta = (\omega - \eta)_{(0)}\eta = \omega_{(0)}\eta - \eta_{(0)}\eta = \eta_{(-2)}\mathbb{1} - \eta_{(-2)}\mathbb{1} = 0$$

and thus  $\eta$  and  $\xi$  are mutually orthogonal by Theorem 5.1 of [FZ92]. We will show that  $\eta$  is the conformal vector of  $\langle A \rangle$ . For this, it suffices to show that  $\eta_{(1)}a = 2a$  and  $\eta_{(0)}a = a_{(-2)}\mathbb{1}$  for  $a \in A$  since  $\eta_{(0)}$  is a derivation on  $V_{\mathbb{R}}$ . Let  $a \in A$ . Then

$$2a = \omega_{(1)}a = (\eta + \xi)_{(1)}a = \eta_{(1)}a + \xi_{(1)}a$$

and we have  $\xi_{(1)}a = 2a - \eta_{(1)}a \in B \cap \mathbb{R}A = 0$ . Therefore  $\eta_{(1)}a = 2a$  and  $\xi_{(1)}a = 0$ . Next we will prove  $\eta_{(0)}a = a_{(-2)}\mathbb{1}$ . Since  $\xi_{(2)}a \in (V_{\mathbb{R}})_1 = 0$ , we have

$$(\xi_{(0)}a | \xi_{(0)}a) = (a | \xi_{(2)}\xi_{(0)}a) = (a | [\xi_{(2)}, \xi_{(0)}]a) = (a | 2\xi_{(1)}a) = 0$$

and  $\xi_{(0)}a = 0$  by the positive definiteness of  $V_{\mathbb{R}}$ . Then  $\eta_{(0)}a = (\omega - \xi)_{(0)}a = \omega_{(0)}a - \xi_{(0)}a = \omega_{(0)}a = a_{(-2)}\mathbb{1}$ . Therefore,  $\eta$  is the conformal vector of  $\langle A \rangle$ .

Finally, we will show that  $\eta \in \mathbb{R}A$  is uniquely determined by the condition  $(\eta | a) = (a | a)$  for all  $a \in A$ . Since  $\eta_{(1)}a = a_{(1)}a = 2a$ , one has

$$2(a | a) = (\eta_{(1)}a | a) = (\eta | a_{(1)}a) = 2(\eta | a)$$

and the conformal vector  $\eta \in \mathbb{R}A$  satisfies the condition. Let  $a_1, \dots, a_n \in A$  be a linear basis of  $\mathbb{R}A$  and write  $\eta = c_1a_1 + \dots + c_na_n$  with  $c_1, \dots, c_n \in \mathbb{R}$ . Then the Gram matrix  $[(a_i | a_j)]_{1 \leq i, j \leq n}$  is non-singular and the condition  $(\eta | a_i) = (a_i | a_i)$  uniquely determines the coefficients  $c_i \in \mathbb{R}$ . Therefore, the conformal vector of  $\langle A \rangle$  is determined by the condition  $\eta \in \mathbb{R}A$  and  $(\eta | a) = (a | a)$  for all  $a \in A$ . ■

**Remark 4.5.** From the proof of Lemma 4.4, one sees that  $\langle A \rangle \subset \text{Com}_{V_{\mathbb{R}}} \langle \xi \rangle = \text{Ker}_{V_{\mathbb{R}}} \xi_{(0)}$  (cf. Theorem 5.1 of [FZ92]). It is shown in Proposition 4 of [Ma05] that  $\mathbb{R}A = \langle A \rangle \cap (V_{\mathbb{R}})_2$  if  $\mathbb{R}A = (V_{\mathbb{R}})_2 \cap \text{Ker}_V \xi_{(1)}$ .

#### 4.1 The case $\langle x, y \rangle \cong U_{2A}$

In this subsection, we determine the Griess algebra of  $\langle a, b, x, y \rangle$  when  $\langle x, y \rangle$  is the 2A-algebra.

**Theorem 4.6.** *Let  $a, b, x, y$  be Ising vectors such that  $\langle a, b \rangle$  is isomorphic to the 3A-algebra,  $\langle x, y \rangle$  is isomorphic to the 2A-algebra and  $(a | x) = (a | y) = (b | x) = (b | y) = 2^{-5}$ . Then the Griess subalgebra of  $\langle a, b, x, y \rangle$  generated by  $a, b, x, y$  is 13-dimensional with a basis*

$$A = \{u = u_{a,b}, a, b, c = \tau_a b, a \circ x, b \circ x, c \circ x, a \circ y, b \circ y, c \circ y, x, y, z = x \circ y\}.$$

The conformal vector of  $\langle a, b, x, y \rangle$  is

$$\eta = \frac{1}{6}u + \frac{16}{27}(a + b + c + a \circ x + b \circ x + c \circ x + a \circ y + b \circ y + c \circ y) + \frac{4}{9}(x + y + z)$$

and has 52/15. The VOA  $\langle a, b, x, y \rangle$  has a full sub VOA isomorphic to

$$L(c_3, 0) \otimes L(c_4, 0) \otimes L(c_5, 0) \otimes L(c_6, 0) = L(4/5, 0) \otimes L(6/7, 0) \otimes L(25/28, 0) \otimes L(11/12, 0).$$

**Proof:** Set  $c = \tau_a b = \tau_b a$  and  $z = x \circ y$ . We have seen in Lemma 3.2 that both  $\langle a, b \circ x \rangle$  and  $\langle a, b \circ y \rangle$  are isomorphic to the 6A-algebra and

$$\begin{aligned}\langle a, b \circ x \rangle_2 &= \text{Span}\{u_{a,b}, x, a, b, c, a \circ x, b \circ x, c \circ x\}, \\ \langle a, b \circ y \rangle_2 &= \text{Span}\{u_{a,b}, y, a, b, c, a \circ y, b \circ y, c \circ y\}.\end{aligned}$$

We have  $u_{a,b} = u_{a \circ x, b \circ x}$  in  $\langle a, b \circ x \rangle$  and  $u_{a,b} = u_{a \circ y, b \circ y}$  in  $\langle a, b \circ y \rangle$ . Therefore

$$u = u_{a,b} = u_{a \circ x, b \circ x} = u_{a \circ y, b \circ y}.$$

It follows from Lemma 2.10 that  $(a \mid x \circ y) = (x \mid a \circ y) = (y \mid a \circ x) = 0$  so that  $\sigma_a(x \circ y) = x \circ y$  and

$$(a \circ x) \circ (a \circ y) = (\sigma_a x) \circ (\sigma_a y) = \sigma_a(x \circ y) = x \circ y.$$

Therefore, the 3-set  $\{a \circ x, a \circ y, x \circ y\}$  forms a 2A-triple. Similarly, the 3-sets  $\{b \circ x, b \circ y, x \circ y\}$  and  $\{c \circ x, c \circ y, x \circ y\}$  also form 2A-triples. By (1) of Proposition 4.3, the Griess algebra of the subalgebra  $\langle a \circ x, b \circ y \rangle$  is isomorphic to the 6A-algebra with a basis

$$\begin{aligned}a \circ x, b \circ y, \tau_{a \circ x}(b \circ y) &= c \circ y, \tau_{b \circ y}(a \circ x) = c \circ x, \tau_{c \circ x}(b \circ y) = a \circ y, \\ \tau_{c \circ y}(a \circ x) &= b \circ x, (a \circ x) \circ (a \circ y) = x \circ y = z, u_{a \circ x, b \circ x} = u_{a \circ y, b \circ y} = u_{a,b}.\end{aligned}$$

Therefore, the linear span of  $A$  forms a subalgebra in the Griess algebra. The determinant of the Gram matrix of  $A$  is  $3^{25} \cdot 11^5 / 2^{81} \cdot 5$  so that this matrix is non-singular. Thus  $A$  is a basis of the Griess subalgebra generated by  $a, b, x$  and  $y$ .

One can directly verify that the vector  $\eta$  satisfies  $(\eta \mid t) = (t \mid \eta)$  for all  $t \in A$ . Therefore,  $\eta$  is the conformal vector of  $\langle a, b, x, y \rangle$  by Lemma 4.4. The central charge of  $\eta$  is  $2(\eta \mid \eta) = 52/15$ . We know that the subalgebra  $\langle a, b, x \rangle$  is isomorphic to the 6A-algebra and has a Virasoro frame  $L(4/5, 0) \otimes L(6/7, 0) \otimes L(25/28, 0)$  (see (8) of Theorem 2.13). By a direct computation, one finds that the commutant subalgebra of  $\langle a, b, x \rangle$  in  $\langle a, b, x, y \rangle$  is generated by the following  $c = 11/12$  Virasoro vector:

$$f = -\frac{5}{24}u - \frac{2}{27}(a+b+c+a \circ x+b \circ x+c \circ x) + \frac{16}{27}(a \circ y+b \circ y+c \circ y) - \frac{1}{18}x + \frac{4}{9}(y+z). \quad (4.1)$$

Therefore,  $\langle a, b, x, y \rangle$  has a unitary Virasoro frame isomorphic to

$$L(c_3, 0) \otimes L(c_4, 0) \otimes L(c_5, 0) \otimes L(c_6, 0) = L(4/5, 0) \otimes L(6/7, 0) \otimes L(25/28, 0) \otimes L(11/12, 0).$$

This completes the proof. ■

## 4.2 The case $\langle x, y \rangle \cong U_{3A}$

Suppose  $\langle a, b \rangle$  and  $\langle x, y \rangle$  are the 3A-algebras and  $(a|x) = (a|y) = (b|x) = (b|y) = 2^{-5}$ . Let  $c = \tau_a b = \tau_b a$  and  $z = \tau_x y = \tau_y x$ . We also recall from (2.8) that  $u_{a,b}$  and  $u_{x,y}$  denote the characteristic  $c = 4/5$  Virasoro vectors in the 3A-algebras  $\langle a, b \rangle$  and  $\langle x, y \rangle$ , respectively. Since all of  $\langle a, b, x \rangle$ ,  $\langle a, b, y \rangle$ ,  $\langle a, b, z \rangle$ ,  $\langle a, x, y \rangle$ ,  $\langle b, x, y \rangle$  and  $\langle c, x, y \rangle$  are isomorphic to the 6A-algebra by Lemma 3.2 and  $U_{6A}$  has the unique characteristic  $c = 4/5$  Virasoro vector, we have

$$u_{a,b} = u_{a \circ x, b \circ x} = u_{a \circ y, b \circ y} = u_{a \circ z, b \circ z}, \quad u_{x,y} = u_{a \circ x, a \circ y} = u_{b \circ x, b \circ y} = u_{c \circ x, c \circ y}. \quad (4.2)$$

Consider the subalgebra generated by  $a \circ x$ ,  $a \circ y$  and  $b \circ x$ . Since  $\tau_{a \circ x} = \tau_a \tau_x$ ,  $\tau_{a \circ y} = \tau_a \tau_y$  and  $\tau_{b \circ x} = \tau_b \tau_x$  by (5) of Theorem 2.11, we have  $\tau_{a \circ x} \tau_{a \circ y} = \tau_x \tau_y$  and  $\tau_{a \circ x} \tau_{b \circ x} = \tau_a \tau_b$  and obtain the following conjugacy.

$$\begin{array}{ccccc} a \circ x & \xrightarrow{\tau_x \tau_y} & a \circ y & \xrightarrow{\tau_x \tau_y} & a \circ z \\ \tau_a \tau_b \downarrow & & \tau_a \tau_b \downarrow & & \tau_a \tau_b \downarrow \\ b \circ x & \xrightarrow{\tau_x \tau_y} & b \circ y & \xrightarrow{\tau_x \tau_y} & b \circ z \\ \tau_a \tau_b \downarrow & & \tau_a \tau_b \downarrow & & \tau_a \tau_b \downarrow \\ c \circ x & \xrightarrow{\tau_x \tau_y} & c \circ y & \xrightarrow{\tau_x \tau_y} & c \circ z \end{array} \quad (4.3)$$

Set  $H = \langle \tau_{a \circ x}, \tau_{a \circ y}, \tau_{b \circ x} \rangle$ . Then  $H \cong 3^2:2$  acts transitively on the the set of 9 Ising vectors in (4.3). By (2) of Proposition 4.3, we see that  $(a \circ x | b \circ y) = 13 \cdot 2^{-10}$  and hence  $\langle a \circ x, b \circ y \rangle$  is also a 3A-algebra. We have  $\tau_{a \circ x}(b \circ y) = \tau_a b \circ \tau_x y = c \circ z$  and  $\{a \circ y, b \circ x, c \circ z\}$  forms the 3A-triple of  $\langle a \circ y, b \circ x \rangle$ . Similarly, by using the conjugacy relations we can verify that each pair of Ising vectors in (4.3) generate a 3A-algebra and we obtain the following four  $H$ -orbits of 3A-triples:

$$\begin{aligned} \mathcal{L}_1 &= \{\{a \circ x, b \circ x, c \circ x\}, \{a \circ y, b \circ y, c \circ y\}, \{a \circ z, b \circ z, c \circ z\}\}, \\ \mathcal{L}_2 &= \{\{a \circ x, a \circ y, a \circ z\}, \{b \circ x, b \circ y, b \circ z\}, \{c \circ x, c \circ y, c \circ z\}\}, \\ \mathcal{L}_3 &= \{\{a \circ x, b \circ y, c \circ z\}, \{a \circ y, b \circ z, c \circ x\}, \{a \circ z, b \circ x, c \circ y\}\}, \\ \mathcal{L}_4 &= \{\{a \circ x, b \circ z, c \circ y\}, \{a \circ y, b \circ x, c \circ z\}, \{a \circ z, b \circ y, c \circ x\}\}. \end{aligned} \quad (4.4)$$

The configuration of 9 points given by Ising vectors in (4.3) together with the lines given by 3-sets forming 3A-triples is isomorphic to the affine plane of order 3 and each  $H$ -orbit  $\mathcal{L}_i$  in (4.4) is the set of parallel lines. By (4.2), all 3A-triples in  $\mathcal{L}_1$  define the same characteristic  $c = 4/5$  Virasoro vector. The sub VOA generated by such 9 Ising vectors is determined in [LSu].

**Lemma 4.7** (Lemma 4.22 and Proposition 4.24 of [LSu]). *For  $1 \leq i \leq 4$ , all 3A-triples in  $\mathcal{L}_i$  define the same characteristic  $c = 4/5$  Virasoro vector  $u^i$ . Moreover, the  $c = 4/5$  Virasoro vectors  $u^i$ ,  $1 \leq i \leq 4$ , are mutually orthogonal.*

As a consequence, we have

$$u_{a \circ x, boy} = u_{a \circ y, boz} = u_{a \circ z, box}, \quad \text{and} \quad u_{a \circ x, boz} = u_{a \circ y, box} = u_{a \circ z, boy}. \quad (4.5)$$

It follows from the relations above that the linear span of the Ising vectors in (4.3) together with the  $c = 4/5$  Virasoro vectors  $u^1 = u_{a,b}$ ,  $u^2 = u_{x,y}$ ,  $u^3 = u_{a \circ x, boy}$  and  $u^4 = u_{a \circ x, boz}$  forms a subalgebra in the Griess algebra. The products and inner products of this Griess subalgebra is described by using the 3A-algebras along with the incidence structure of the affine plane of order 3. More precisely, the subalgebra generated by Ising vectors in (4.3) is described as follows.

**Theorem 4.8** (Lemma 3.8 of [HLY12b] and Theorem 4.25 of [LSu]). *The sub VOA generated by 9 Ising vectors in (4.3) is isomorphic the ternary code VOA  $M_C$  constructed in [KMY00]. It is actually generated by 3 Ising vectors  $a \circ x$ ,  $a \circ y$  and  $b \circ x$  and its Griess algebra is 12-dimensional spanned by 9 Ising vectors in (4.3) and 4 mutually orthogonal  $c = 4/5$  Virasoro vectors  $u_{a,b}$ ,  $u_{x,y}$ ,  $u_{a \circ x, boy}$  and  $u_{a \circ x, boz}$  with one linear relation*

$$\begin{aligned} & u_{a,b} + u_{x,y} + u_{a \circ x, boy} + u_{a \circ x, boz} \\ &= \frac{32}{45}(a \circ x + b \circ x + c \circ x + a \circ y + b \circ y + c \circ y + a \circ z + b \circ z + c \circ z). \end{aligned}$$

Moreover,  $\omega = u_{a,b} + u_{x,y} + u_{a \circ x, boy} + u_{a \circ x, boz}$  gives the conformal vector of the subalgebra.

Set

$$A = \{u_{a,b}, u_{x,y}, u_{a \circ x, boy}, u_{a \circ x, boz}, a, b, c = \tau_a b, x, y, z = \tau_x y, a \circ x, a \circ y, a \circ z, b \circ x, b \circ y, c \circ z, c \circ x, c \circ y, c \circ z\}. \quad (4.6)$$

In the rest of this subsection, we will prove that the linear span of  $A$  forms a subalgebra in the Griess algebra. By Lemma 3.2, the following subalgebras are isomorphic to the Griess algebra of the 6A-algebra:

$$\begin{aligned} \langle a, b, x \rangle_2 &= \text{Span}\{u_{a,b}, x, a, b, c, a \circ x, b \circ x, c \circ x\}, \\ \langle a, b, y \rangle_2 &= \text{Span}\{u_{a,b}, y, a, b, c, a \circ y, b \circ y, c \circ y\}, \\ \langle a, b, z \rangle_2 &= \text{Span}\{u_{a,b}, z, a, b, c, a \circ z, b \circ z, c \circ z\}, \\ \langle a, x, y \rangle_2 &= \text{Span}\{u_{x,y}, a, x, y, z, a \circ x, a \circ y, a \circ z\}, \\ \langle b, x, y \rangle_2 &= \text{Span}\{u_{x,y}, b, x, y, z, b \circ x, b \circ y, b \circ z\}, \\ \langle c, x, y \rangle_2 &= \text{Span}\{u_{x,y}, c, x, y, z, c \circ x, c \circ y, c \circ z\}. \end{aligned} \quad (4.7)$$

Therefore, it is enough to show that the products of  $a, b, c, x, y, z$  and  $u_{a \circ x, boy}$ ,  $u_{a \circ x, boz}$  lie in the linear span of  $A$  and determine the inner products among them. Set  $G = \langle \tau_a, \tau_b, \tau_x, \tau_y \rangle$ . Then the sets of 3A-triples  $\mathcal{L}_1$  and  $\mathcal{L}_2$  in (4.4) are  $G$ -stable, whereas  $\mathcal{L}_3$  and  $\mathcal{L}_4$  are  $H$ -stable but not  $G$ -stable. We have the following conjugacy.

**Lemma 4.9.** For  $e \in \{a, b, c, x, y, z\}$ , we have  $\tau_e u_{a \circ x, b \circ y} = u_{a \circ x, b \circ z}$ .

**Proof:** Let  $e \in \{a, b, c, x, y, z\}$ . Then we have  $\tau_e \mathcal{L}_3 = \mathcal{L}_4$  and the conjugacy  $\tau_e u_{a \circ x, b \circ y} = u_{a \circ x, b \circ z}$  of the characteristic  $c = 4/5$  Virasoro vectors follows from that of the corresponding 3A-triples.  $\blacksquare$

**Lemma 4.10.** Let  $e \in \{x, y, z, a, b, c\}$ . Then

$$(e \mid u_{a \circ x, b \circ y}) = (e \mid u_{a \circ x, b \circ z}) = \frac{1}{80}.$$

**Proof:** Recall that

$$u_{a \circ x, b \circ y} = \frac{2^6}{135} (2a \circ x + 2b \circ y + c \circ z - 16(a \circ x)_{(1)}(b \circ y)).$$

First we have

$$\begin{aligned} (a \mid (a \circ x)_{(1)}(b \circ y)) &= (a_{(1)}(a \circ x) \mid b \circ y) = \frac{1}{4} (a + a \circ x - x \mid b \circ y) \\ &= \frac{1}{4} \left( \frac{5}{2^{10}} + \frac{13}{2^{10}} - \frac{5}{2^{10}} \right) = \frac{13}{2^{12}}. \end{aligned}$$

Hence,

$$(a \mid u_{a \circ x, b \circ y}) = \frac{2^6}{135} \left( 2 \cdot \frac{1}{2^5} + 2 \cdot \frac{5}{2^{10}} + \frac{5}{2^{10}} - 16 \cdot \frac{13}{2^{12}} \right) = \frac{1}{80}.$$

The other cases can be proved similarly.  $\blacksquare$

Let us compute  $e_{(1)}u$  for  $e \in \{a, b, c, x, y, z\}$  and  $u \in \{u_{a \circ x, b \circ y}, u_{a \circ x, b \circ z}\}$ . Recall that  $a \circ x = 2^{-2}(a + x - a_{(1)}x)$  in (2.7). For simplicity, we regard  $a \circ x$  as a bilinear product and define  $a \circ (x + y + z) = a \circ x + a \circ y + a \circ z$ .

**Lemma 4.11.** Let  $u = u_{a \circ x, b \circ y}$  or  $u_{a \circ x, b \circ z}$ . Then we have

$$\begin{aligned} \sigma_e(u + \tau_e u) &= -\frac{1}{2}u_{a,b} - u_{x,y} + \frac{4}{15}e - \frac{8}{45}e \circ (x + y + z) \\ &\quad - \frac{16}{45}(a + b + c) + \frac{8}{15}(x + y + z) + \frac{32}{45}(a + b + c) \circ (x + y + z) \end{aligned}$$

if  $e \in \{a, b, c\}$ , and

$$\begin{aligned} \sigma_e(u + \tau_e u) &= -u_{a,b} - \frac{1}{2}u_{x,y} + \frac{4}{15}e - \frac{8}{15}(a + b + c) \circ e \\ &\quad + \frac{8}{15}(a + b + c) - \frac{16}{45}(x + y + z) + \frac{32}{45}(a + b + c) \circ (x + y + z) \end{aligned}$$

if  $e \in \{x, y, z\}$ .



**Proof:** We only compute the case  $e = x$ . The case  $e = a$  is similar. By (2.8), we have

$$\begin{aligned} u_{a \circ x, b \circ y} &= \frac{2^6}{135} (2a \circ x + 2b \circ y + c \circ z - 16(a \circ x)_{(1)}(b \circ y)), \\ u_{a \circ x, b \circ z} &= \frac{2^6}{135} (2a \circ x + 2b \circ z + c \circ y - 16(a \circ x)_{(1)}(b \circ z)). \end{aligned}$$

Since  $\sigma_x(a \circ x) = a$  and  $u + \tau_x u = u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z}$ , we have

$$\begin{aligned} \sigma_x(u + \tau_x u) &= \sigma_x(u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z}) \\ &= \frac{2^6}{135} \sigma_x(4a \circ x + 2b \circ y + 2b \circ z + c \circ y + c \circ z - 16(a \circ x)_{(1)}(b \circ y + b \circ z)) \\ &= \frac{2^6}{135} (4a + 2\sigma_x(b \circ y + b \circ z) + \sigma_x(c \circ y + c \circ z) - 16a_{(1)}\sigma_x(b \circ y + b \circ z)). \end{aligned} \quad (4.8)$$

By Lemma 2.14 (cf. Eq. (2.10)), we have

$$\begin{aligned} \sigma_x(b \circ y + b \circ z) &= -\frac{45}{2^7} u_{x,y} - \frac{1}{2^2} b + \frac{1}{2^4} x + \frac{1}{2^2} b \circ x + \frac{1}{2^2} (y + z) + b \circ y + b \circ z, \\ \sigma_x(c \circ y + c \circ z) &= -\frac{45}{2^7} u_{x,y} - \frac{1}{2^2} c + \frac{1}{2^4} x + \frac{1}{2^2} c \circ x + \frac{1}{2^2} (y + z) + c \circ y + c \circ z. \end{aligned} \quad (4.9)$$

By using the Griess algebras of  $\langle a, b \circ y \rangle$  and  $\langle a, b \circ z \rangle$  (cf. (7) of Theorem 2.13), we have

$$\begin{aligned} &a_{(1)} \sigma_x(b \circ y + b \circ z) \\ &= a_{(1)} \left( -\frac{45}{2^7} u_{x,y} - \frac{1}{2^2} b + \frac{1}{2^4} x + \frac{1}{2^2} b \circ x + \frac{1}{2^2} (y + z) + b \circ y + b \circ z \right) \\ &= -\frac{45}{2^7} a_{(1)} u_{x,y} - \frac{1}{2^2} \left( -\frac{135}{2^{10}} u_{a,b} + \frac{1}{2^4} (2a + 2b + c) \right) + \frac{1}{2^4} \cdot \frac{1}{4} (a + x - a \circ x) \\ &\quad + \frac{1}{2^2} \left( \frac{45}{2^{10}} u_{a,b} + \frac{1}{2^5} (x + a + b \circ x - b - c - a \circ x - c \circ x) \right) \\ &\quad + \frac{1}{2^2} \left( \frac{1}{4} (a + y - a \circ y) + \frac{1}{4} (a + z - a \circ z) \right) \\ &\quad + \frac{45}{2^{10}} u_{a,b} + \frac{1}{2^5} (y + a + b \circ y - b - c - a \circ y - c \circ y) \\ &\quad + \frac{45}{2^{10}} u_{a,b} + \frac{1}{2^5} (z + a + b \circ z - b - c - a \circ z - c \circ z). \end{aligned} \quad (4.10)$$

Note that  $a_{(1)} u_{x,y} = 0$  in the 6A-algebra  $\langle a, x, y \rangle$  and thus by plugging (4.9) and (4.10) into (4.8), we have

$$\begin{aligned} \sigma_x(u + \tau_x u) &= \frac{8}{15} (a + b + c) - \frac{4}{45} x - \frac{16}{45} (y + z) + \frac{8}{45} (a \circ x + b \circ x + c \circ x) \\ &\quad + \frac{32}{45} (a \circ y + b \circ y + c \circ y + a \circ z + b \circ z + c \circ z) - u_{a,b} - \frac{1}{2} u_{x,y} \end{aligned}$$

as desired. ■

**Lemma 4.12.** *Let  $u = u_{a \circ x, boy}$  or  $u_{a \circ x, boz}$ . Then we have*

$$\begin{aligned} e_{(1)}u &= \frac{1}{16}u_{a,b} + \frac{1}{8}u_{x,y} + \frac{5}{32}u + \frac{3}{32}\tau_e u + \frac{1}{15}e + \frac{1}{15}e \circ (x + y + z) \\ &\quad + \frac{2}{45}(a + b + c) - \frac{1}{15}(x + y + z) - \frac{4}{45}(a + b + c) \circ (x + y + z) \end{aligned}$$

if  $e \in \{a, b, c\}$ , and

$$\begin{aligned} e_{(1)}u &= \frac{1}{8}u_{a,b} + \frac{1}{16}u_{x,y} + \frac{5}{32}u + \frac{3}{32}\tau_e u + \frac{1}{15}e + \frac{1}{15}(a + b + c) \circ e \\ &\quad - \frac{1}{15}(a + b + c) + \frac{2}{45}(x + y + z) - \frac{4}{45}(a + b + c) \circ (x + y + z) \end{aligned}$$

if  $e \in \{x, y, z\}$ .

**Proof:** Again we only compute the case  $e = x$ . By (2.5), we have

$$x_{(1)}u = 8(x | u)x + \frac{5}{32}u + \frac{3}{32}\tau_x u - \frac{1}{8}\sigma_x(u + \tau_x u).$$

Then the lemma immediately follows from Lemmas 4.10 and 4.11. ■

Recall the set  $A$  of Virasoro vectors in (4.6). By Lemma 4.12 and  $H$ -invariance of  $u_{a \circ x, boy}$  and  $u_{a \circ x, boz}$ , the products  $e_{(1)}u$  for  $e \in \{a, b, c, x, y, z\}$  and  $u \in \{u_{a \circ x, boy}, u_{a \circ x, boz}\}$  are completely determined by linear combinations of the Virasoro vectors in  $A$ . Therefore, the linear span of  $A$  forms a subalgebra of the Griess algebra of  $\langle a, b, x, y \rangle$ .

Next we determine the conformal vector of  $\langle a, b, x, y \rangle$ .

**Proposition 4.13.** *Set*

$$\eta = \frac{17}{22}(u_{a,b} + u_{x,y}) + \frac{10}{11}(u_{a \circ x, boy} + u_{a \circ x, boz}) + \frac{16}{33}(a + b + c + x + y + z).$$

Then  $(\eta | t) = (t | t)$  for all  $t \in A$  and therefore  $\eta$  is the conformal vector of  $\langle a, b, x, y \rangle$  of central charge  $228/55$ .

**Proof:** Since  $u_{a,b}$ ,  $u_{x,y}$ ,  $u_{a \circ x, boy}$ ,  $u_{a \circ x, boz}$  are mutually orthogonal by Lemma 4.7, it is straightforward to verify  $(\eta | t) = (t | t)$  for all  $t \in A$  by using the 6A-algebras given in (4.7) and Lemma 4.10. By Lemma 4.4,  $\eta$  is the conformal vector of  $\langle a, b, x, y \rangle$ . The central charge is given by

$$\begin{aligned} 2(\eta | \eta) &= \frac{17}{22}(\eta | u_{a,b} + u_{x,y}) + \frac{10}{11}(\eta | u_{a \circ x, boy} + u_{a \circ x, boz}) + \frac{16}{33}(\eta | a + b + c + x + y + z) \\ &= 2 \cdot \left( \frac{17}{22} \cdot \frac{2}{5} \cdot 2 + \frac{10}{11} \cdot \frac{2}{5} \cdot 2 + \frac{16}{33} \cdot \frac{1}{4} \cdot 6 \right) = \frac{228}{55} \end{aligned}$$

as claimed. ■

Summarizing everything, the structure of the Griess algebra of  $\langle a, b, x, y \rangle$  is described as follows.

**Theorem 4.14.** *Let  $a, b, x, y$  be Ising vectors of  $V_{\mathbb{R}}$  such that  $\langle a, b \rangle$  and  $\langle x, y \rangle$  are isomorphic to the 3A-algebra and  $(a|x) = (a|y) = (b|x) = (b|y) = 2^{-5}$ . Then the Griess subalgebra of  $\langle a, b, x, y \rangle$  generated by  $a, b, x$  and  $y$  is 18-dimensional spanned by*

$$A = \{u_{a,b}, u_{x,y}, u_{a \circ x, b \circ y}, u_{a \circ x, b \circ z}, a, b, c = \tau_a b, x, y, z = \tau_x y, \\ a \circ x, a \circ y, a \circ z, b \circ x, b \circ y, c \circ z, c \circ x, c \circ y, c \circ z\}$$

with one linear relation

$$u_{a,b} + u_{x,y} + u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z} = \frac{32}{45}(a \circ x + a \circ y + a \circ z + b \circ x + b \circ y + c \circ z, c \circ x, c \circ y, c \circ z).$$

The conformal vector of  $\langle a, b, x, y \rangle$  is

$$\eta = \frac{17}{22}(u_{a,b} + u_{x,y}) + \frac{10}{11}(u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z}) + \frac{16}{33}(a + b + c + x + y + z)$$

and has central charge  $228/55$ . The VOA  $\langle a, b, x, y \rangle$  has a full sub VOA isomorphic to

$$L(c_3, 0) \otimes L(c_3, 0) \otimes L(c_3, 0) \otimes L(c_3, 0) \otimes L(c_8, 0) = L(4/5, 0)^{\otimes 4} \otimes L(52/55, 0).$$

**Proof:** We have already shown that the Griess subalgebra generated by  $a, b, x, y$  is linearly spanned by  $A$ . We also know the linear relation in Theorem 4.8. It is straightforward to verify that the determinant of the Gram matrix of  $A \setminus \{u_{a \circ x, b \circ z}\}$  is equal to  $3^{52} \cdot 11 \cdot 13^6 / 2^{138} \cdot 5^3$ . Therefore, the linear span of  $A$  is 18-dimensional. The conformal vector of  $\langle a, b, x, y \rangle$  is given in Proposition 4.13.

The subalgebra  $\langle a \circ x, a \circ y, b \circ x \rangle$  is isomorphic to the ternary code VOA and has the Virasoro frame  $u_{a,b} + u_{x,y} + u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z}$ . That is,  $\langle a \circ x, a \circ y, b \circ x \rangle$  is a  $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ -graded simple current extension of  $L(4/5, 0)^{\otimes 4}$ . Then the element

$$\begin{aligned} \xi &= \eta - (u_{a,b} + u_{x,y} + u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z}) \\ &= \frac{5}{22}(u_{a,b} + u_{x,y}) + \frac{1}{11}(u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z}) + \frac{16}{33}(a + b + c + x + y + z) \end{aligned} \quad (4.11)$$

is a Virasoro vector orthogonal to  $u_{a,b} + u_{x,y} + u_{a \circ x, b \circ y} + u_{a \circ x, b \circ z}$ . The central charge of  $\xi$  is  $228/55 - 4 \cdot 4/5 = 52/55 = c_8$ . Therefore,  $\langle a, b, x, y \rangle$  has the full sub VOA

$$\langle a \circ x, a \circ y, b \circ x, \xi \rangle \cong \langle a \circ x, a \circ y, b \circ x \rangle \otimes \langle \xi \rangle$$

which contains a unitary Virasoro frame  $L(4/5, 0)^{\otimes 4} \otimes L(52/55, 0)$ . ■

### 4.3 The (2A,3A)-generated subalgebras

Let  $a, b$  be Ising vectors of  $V_{\mathbb{R}}$  such that  $(a|b) = 13 \cdot 2^{-10}$  and let  $I_{a,b}$  be defined as in (3.1). We will choose  $x^i \in I_{a,b}$  and define subalgebras  $X^{[i]} = \langle X^{[i-1]}, x^i \rangle$  inductively as follows.

**Definition 4.15.** Set  $X^{[0]} = \langle a, b \rangle$ . Suppose we have chosen  $x^1, \dots, x^i \in I_{a,b}$  and defined

$$X^{[i]} := \langle a, b, x^1, \dots, x^i \rangle \quad \text{for some } i \geq 0. \quad (4.12)$$

Then choose  $x^{i+1} \in I_{a,b}$  such that  $x^{i+1} \notin X^{[i]}$  and  $(x^{i+1} | x^j) = 2^{-5}$  for all  $1 \leq j \leq i$ . The algebra  $X^{[i+1]}$  is defined by

$$X^{[i+1]} := \langle X^{[i]}, x^{i+1} \rangle = \langle a, b, x^1, \dots, x^i, x^{i+1} \rangle.$$

We will show that the structure of the Griess algebra of  $X^{[n]}$  does not depend on the choices of  $x^i$ ,  $1 \leq i \leq n$ , and is uniquely determined up to isomorphism. First, we observe from [Ma05] that the Griess algebra of  $\langle x^1, \dots, x^n \rangle$  is uniquely determined by our choice.

**Proposition 4.16** ([Ma05]). *Suppose  $x^1, \dots, x^n$  are Ising vectors such that  $(x^i | x^j) = 2^{-5}$  and  $x^k \notin \langle x^1, \dots, x^{k-1} \rangle$  for any  $1 \leq i < j \leq n$  and  $1 < k \leq n$ . Then the Griess algebra of  $\langle x^1, \dots, x^n \rangle$  has a unique structure with a basis  $\{x^i, x^j \circ x^k \mid 1 \leq i \leq n, 1 \leq j < k \leq n\}$ .*

**Proof:** Let  $i, j, k$  be distinct. By Lemma 2.10, we have  $(x^i \circ x^j | x^k) = 0$  and  $\langle x^i \circ x^j, x^k \rangle$  is the 2B-algebra. Therefore  $x_{(1)}^k (x^i \circ x^j) = 0$  and  $\sigma_{x^k}$  fixes  $x^i \circ x^j$ . Then by conjugating  $\{x^i, x^j, x^i \circ x^j\}$  by  $\sigma_{x^k}$  we see that  $\{x^i \circ x^j, x^i \circ x^k, x^j \circ x^k\}$  forms a 2A-triple. Moreover, if  $l$  is distinct from  $i, j$  and  $k$ , then

$$(x^i \circ x^j | x^k \circ x^l) = (x^i + x^j - 4x_{(1)}^i x^j | x^k \circ x^l) = -4 \left( x^i | x_{(1)}^j (x^k \circ x^l) \right) = 0.$$

Therefore,  $\langle x^i \circ x^j, x^k \circ x^l \rangle$  is a 2B-algebra. Hence, the Griess subalgebra generated by  $x^i$ ,  $1 \leq i \leq n$ , is uniquely determined and linearly spanned by Ising vectors  $x^i$ ,  $1 \leq i \leq n$  and  $x^j \circ x^k$ ,  $1 \leq j < k \leq n$ . Such Ising vectors can be realized inside  $V_{\sqrt{2}A_n}^+$  and known to be linearly independent (cf. [DLMN98, LSY07, Ma01]). This completes the proof. ■

**Remark 4.17.** The Griess algebra of  $\langle x^1, \dots, x^n \rangle$  coincides with that of  $M_{A_n}$  in [LSY07] (cf. Proposition A.3). Set  $\tilde{x}^1 = x^1$  and  $\tilde{x}^i = x^i \circ x^{i-1}$  for  $2 \leq i \leq n$ . It is clear that  $\langle x^1, \dots, x^n \rangle = \langle \tilde{x}^1, \dots, \tilde{x}^n \rangle$ . We have  $(\tilde{x}^i | \tilde{x}^j) = 0$  if  $|i - j| > 1$  and  $(\tilde{x}^k | \tilde{x}^{k+1}) = 2^{-5}$  for  $1 \leq k < n$ . Then the associated involutions  $\sigma_{\tilde{x}^1}, \dots, \sigma_{\tilde{x}^n}$  acting on the subalgebra  $\langle \tilde{x}^1, \dots, \tilde{x}^n \rangle$  satisfy the Coxeter relation of type  $A_n$  and hence  $\langle \sigma_{\tilde{x}^1}, \dots, \sigma_{\tilde{x}^n} \rangle$  is isomorphic to the symmetric group  $S_{n+1}$ . If we identify this group as a permutation group of the  $(n+1)$ -set  $\{0, 1, 2, \dots, n\}$ , then the involutions  $\sigma_{x^i}$  and  $\sigma_{x^j \circ x^k} = \sigma_{x^j} \sigma_{x^k} \sigma_{x^j}$  correspond to the transpositions  $(0 \ i)$  and  $(j \ k)$ , respectively.

**Theorem 4.18.** *With reference to the above, the Griess algebra generated by Ising vectors  $a, b, x^1, \dots, x^n$  is uniquely determined and has the following basis:*

$$u = u_{a,b}, \ a, \ b, \ c = \tau_a b, \ x^i, \ a \circ x^i, \ b \circ x^i, \ c \circ x^i, \ x^j \circ x^k, \ 1 \leq i \leq n, \ 1 \leq j < k \leq n.$$

The conformal vector of  $X^{[n]} = \langle a, b, x^1, \dots, x^n \rangle$  is given by

$$\begin{aligned} \omega^n = & \frac{3(3-n)}{2(n+7)}u + \frac{16}{3(n+7)} \left( a + b + c + \sum_{i=1}^n (a \circ x^i + b \circ x^i + c \circ x^i) \right) \\ & + \frac{4}{n+7} \left( \sum_{i=1}^n x^i + \sum_{1 \leq j < k \leq n} x^j \circ x^k \right) \end{aligned} \quad (4.13)$$

and its central charge is equal to  $(n+2)(5n+29)/5(n+7)$ . The commutant  $\text{Com}_{X^{[n]}} X^{[n-1]}$  has the  $c = c_{n+4}$  conformal vector

$$f^n = \omega^n - \omega^{n-1}. \quad (4.14)$$

Therefore,  $X^{[n]}$  has a unitary Virasoro frame  $L(c_3, 0) \otimes L(c_4, 0) \otimes \dots \otimes L(c_{n+4}, 0)$ .

**Proof:** Set  $c = \tau_a b = \tau_b a$  and

$$A = \{u_{a,b}, a, b, c, x^i, a \circ x^i, b \circ x^i, c \circ x^i, x^j \circ x^k \mid 1 \leq i \leq n, 1 \leq j < k \leq n\}. \quad (4.15)$$

It follows from Lemma 3.2 and (1) of Proposition 4.3 that both  $\langle a, b \circ x^i \rangle$  and  $\langle a \circ x^i, b \circ x^j \rangle$  are isomorphic to the 6A-algebra. Therefore, we have the following subalgebras in the Griess algebra:

$$\begin{aligned} \langle a, b \rangle_2 &= \text{Span}\{u_{a,b}, a, b, c\}, \\ \langle a, a \circ x^i \rangle_2 &= \langle a, x^i \rangle_2 = \text{Span}\{a, a \circ x^i, x^i\}, \\ \langle a, b \circ x^i \rangle_2 &= \text{Span}\{u_{a,b}, x^i, a, b, c, a \circ x^i, b \circ x^i, c \circ x^i\}, \\ \langle a \circ x^i, b \circ x^i \rangle_2 &= \sigma_{x^i} \langle a, b \rangle_2 = \text{Span}\{u_{a,b}, a \circ x^i, b \circ x^i, c \circ x^i\}, \\ \langle a \circ x^i, b \circ x^j \rangle_2 &= \text{Span}\{u_{a,b}, x^i \circ x^j, a \circ x^i, b \circ x^i, c \circ x^i, a \circ x^j, b \circ x^j, c \circ x^j\}. \end{aligned}$$

In particular, the following orthogonality relation holds.

$$(u_{a,b} \mid x^i) = (u_{a,b} \mid x^i \circ x^j) = 0.$$

Applying Proposition 4.16 to the subalgebra  $\langle a, x^1, \dots, x^n \rangle$ , we also have the following orthogonality.

$$(a \mid x^j \circ x^k) = (a \circ x^i \mid x^j) = (x^i \mid x^j \circ x^k) = (a \circ x^i \mid x^j \circ x^k) = 0.$$

On  $X^{[n]}$ , all  $\tau_{x^i}$  and  $\tau_{x^i \circ x^j}$  are trivial and we have  $\tau_{e \circ x^i} = \tau_e$  for  $e \in \{a, b, c\}$  by (5) of Theorem 2.9. Then  $\langle \tau_a, \tau_b \rangle \cong S_3$  acts faithfully on each of the 3-sets  $\{a \circ x^i, b \circ x^i, c \circ x^i\}$ ,  $1 \leq i \leq n$ . The group generated by  $\sigma_{x^i}$ ,  $1 \leq i \leq n$ , is isomorphic to  $S_{n+1}$  and this group acts transitively on the set of Ising vectors  $\{x^i, x^j \circ x^k \mid 1 \leq i \leq n, 1 \leq j < k \leq n\}$  (cf. Remark

4.17). Set  $G = \langle \tau_a, \tau_b, \sigma_{x^1}, \dots, \sigma_{x^n} \rangle \subset \text{Aut}(X^{[n]})$ . Then  $A$  is  $G$ -invariant and the products and inner products of vectors in  $A$  are uniquely determined in the linear span of  $A$ . Therefore, the linear span of  $A$  forms a subalgebra in the Griess algebra. That  $A$  is linearly independent will be shown in Appendix A.3 using an explicit construction.

Next we prove that  $\omega^n$  is the conformal vector of  $X^{[n]}$ . It is clear that  $\omega^n$  is fixed by  $G$ . By using  $G$ -invariance, one can directly verify that  $\omega^n$  satisfies  $(\omega^n | t) = (t | t)$  for all  $t \in A$ . Therefore,  $\omega^n$  is the conformal vector of  $X^{[n]}$  by Lemma 4.4. The central charge of  $\omega^n$  is  $2(\omega^n | \omega^n) = (n+2)(5n+29)/5(n+7)$ . Since  $X^{[n-1]}$  is a subalgebra of  $X^{[n]}$ , both  $\omega^{n-1}$  and  $f^n = \omega^n - \omega^{n-1}$  are mutually orthogonal Virasoro vectors of  $X^{[n]}$  and  $f^n$  is the conformal vector of  $\text{Com}_{X^{[n]}} X^{[n-1]}$  by Theorem 5.1 of [FZ92]. The central charge of  $f^n$  is

$$2(\omega^n | \omega^n) - 2(\omega^{n-1} | \omega^{n-1}) = \frac{n^2 + 13n + 36}{(n+6)(n+7)} = c_{n+4}.$$

Set

$$v_{a,b} = -\frac{5}{14}u_{a,b} + \frac{16}{21}(a+b+c). \quad (4.16)$$

Then  $v_{a,b}$  is a  $c = c_4 = 6/7$  Virasoro vector and the conformal vector  $\omega^0$  of  $X^{[0]} = \langle a, b \rangle$  is an orthogonal sum  $\omega^0 = u_{a,b} \dot{+} v_{a,b}$  by (6) of Theorem 2.11. Therefore, we have the following orthogonal decompositions:

$$\omega^n = \omega^0 \dot{+} f^1 \dot{+} \dots \dot{+} f^n = u_{a,b} \dot{+} v_{a,b} \dot{+} f^1 \dot{+} \dots \dot{+} f^n. \quad (4.17)$$

This shows  $X^{[n]}$  has a full sub VOA isomorphic to  $L(c_3, 0) \otimes L(c_4, 0) \otimes \dots \otimes L(c_{n+4}, 0)$ . This completes the proof.  $\blacksquare$

Set  $D^{[0]} = \{\tau_x \in \text{Aut}(V) \mid x \in I_{a,b}\}$  and inductively we define

$$D^{[i]} := \{\tau_y \in D^{[i-1]} \mid \tau_y \tau_{x^i} = \tau_{x^i} \tau_y\} \quad (4.18)$$

It is clear that  $\langle D^{[i]} \rangle$  is a subgroup of the centralizer of  $\langle \tau_a, \tau_b, \tau_{x^1}, \dots, \tau_{x^n} \rangle$  in  $\text{Aut}(V)$ . For inductive arguments in the next section, we will consider the action of  $\langle D^{[i]} \rangle$  restricted to a smaller subalgebra.

**Lemma 4.19.** *Each involution in  $D^{[n]}$  acts trivially on  $X^{[n]}$ .*

**Proof:** We prove this by induction on  $n$ . By definition, both  $\langle a, x \rangle$  and  $\langle b, x \rangle$  are 2A-algebras for each  $x \in D^{[0]} = I_{a,b}$ , and it follows that  $\tau_x$  fixes both  $a$  and  $b$ . Thus  $D^{[0]}$  acts trivially on  $X^{[0]} = \langle a, b \rangle$ . Suppose each involution of  $D^{[i]}$  acts on  $X^{[i]} = \langle a, b, x^1, \dots, x^i \rangle$  trivially. Let  $\tau_y \in D^{[i+1]}$ . Then  $\tau_y \in D^{[i]}$  and  $\tau_y$  acts trivially on  $X^{[i]}$ . Since  $\tau_y$  and  $\tau_{x^{i+1}}$  commute, it follows from Theorem 2.6 that  $\langle y, x^{i+1} \rangle$  is a dihedral algebra of type 1A, 2A or 2B. In each case  $\tau_y$  fixes  $x^{i+1}$ . Therefore,  $\tau_y$  acts trivially on  $X^{[i+1]} = \langle X^{[i]}, x^{i+1} \rangle$ .  $\blacksquare$

By this lemma, we have a group homomorphism by restriction.

$$\begin{aligned} \varphi^{[i]} : \langle D^{[i]} \rangle &\longrightarrow \text{Aut}(\text{Com}_V X^{[i]}) \\ g &\longmapsto g|_{\text{Com}_V X^{[i]}} \end{aligned} \quad (4.19)$$

Set  $G^{[i]} := \text{Im } \varphi^{[i]} \subset \text{Aut}(\text{Com}_V X^{[i]})$ . It follows from Corollary 3.7 that  $(G^{[i]}, \varphi^{[i]}(D^{[i]}))$  is a 3-transposition group. Since  $X^{[n-1]} \subset X^{[n]}$  implies  $\text{Com}_V X^{[n-1]} \supset \text{Com}_V X^{[n]}$ , we obtain the following inductive structure:

$$\begin{array}{ccccccc} X^{[0]} & \subset & X^{[1]} & \subset & X^{[2]} & \subset & X^{[3]} & \subset & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \text{Com}_V X^{[0]} & \supset & \text{Com}_V X^{[1]} & \supset & \text{Com}_V X^{[2]} & \supset & \text{Com}_V X^{[3]} & \supset & \dots \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ G^{[0]} & & G^{[1]} & & G^{[2]} & & G^{[3]} & & \dots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \langle D^{[0]} \rangle & \supset & \langle D^{[1]} \rangle & \supset & \langle D^{[2]} \rangle & \supset & \langle D^{[3]} \rangle & \supset & \dots \end{array} \quad (4.20)$$

Note that  $\varphi^{[n-1]}(\tau_{x^n}) \in G^{[n-1]}$  is an external automorphism of  $\text{Com}_V X^{[n-1]}$  in the sense that  $x^n \notin \text{Com}_V X^{[n-1]}$ . By Theorem 5.2 of [FZ92] and (4.17) we have

$$\text{Com}_V X^{[n]} = \text{Com}_V \langle u_{a,b}, v_{a,b}, f^1, \dots, f^n \rangle.$$

Since the decomposition in (4.17) is orthogonal, we have  $f^n \in \text{Com}_V X^{[n-1]}$ . We prove that  $\varphi^{[n-1]}(\tau_{x^n})$  can be described as an internal automorphism defined by  $f^n$ .

**Theorem 4.20.** *Let  $X^{[n]} = \langle a, b, x^1, \dots, x^n \rangle$  be the subalgebra of a VOA  $V$  defined by (4.12) and let  $\omega^n = u_{a,b} \dot{+} v_{a,b} \dot{+} f^1 \dot{+} \dots \dot{+} f^n$  be the orthogonal decomposition of the conformal vector of  $X^{[n]}$  as in (4.17).*

(1) *If  $n$  is odd, then  $\tau_{f^n} = \tau_{x^1} \tau_{x^2} \dots \tau_{x^n}$  in  $\text{Aut}(V)$  and  $\tau_{f^n} = \varphi^{[n-1]}(\tau_{x^n})$  as an automorphism of  $\text{Com}_V X^{[n-1]}$ .*

(2) *If  $n > 0$  is even, then  $\tau_{f^n} = \tau_{f^{n-1}} = \tau_{x^1} \tau_{x^2} \dots \tau_{x^{n-1}}$  in  $\text{Aut}(V)$  and  $\varphi^{[n-1]}(\tau_{f^n})$  is trivial in  $\text{Aut}(\text{Com}_V X^{[n-1]})$ . In this case,  $f^n$  is of  $\sigma$ -type on  $\text{Com}_V X^{[n-1]}$  and satisfies  $\sigma_{f^n} = \varphi^{[n-1]}(\tau_{x^n})$  as an automorphism of  $\text{Com}_V X^{[n-1]}$ .*

**Proof:** By direct computations, we have the following relations in  $X^{[n]}$ .

$$\begin{aligned} a_{(1)} u_{a,b} &= \frac{2}{3} a + \frac{5}{24} u_{a,b} - \frac{7}{24} v_{a,b}, & a_{(1)} v_{a,b} &= \frac{4}{3} a - \frac{5}{24} u_{a,b} + \frac{7}{24} v_{a,b}, \\ x^1_{(1)} v_{a,b} &= \frac{5}{7} x^1 + \frac{3}{14} v_{a,b} - \frac{2}{7} f^1, & x^1_{(1)} f^1 &= \frac{9}{7} x^1 - \frac{3}{14} v_{a,b} + \frac{2}{7} f^1, \\ (x^i \circ x^{i+1})_{(1)} f^i &= \frac{i+5}{i+7} x^i \circ x^{i+1} + \frac{i+6}{4(i+7)} f^i - \frac{i+8}{4(i+7)} f^{i+1}, \\ (x^i \circ x^{i+1})_{(1)} f^{i+1} &= \frac{i+9}{i+7} x^i \circ x^{i+1} - \frac{i+6}{4(i+7)} f^i + \frac{i+8}{4(i+7)} f^{i+1}, \end{aligned} \quad (4.21)$$



where  $1 \leq i \leq n-1$ . By these relations, it follows from Proposition 4.2 of [LY16] that the subalgebras  $\langle a, u_{a,b}, v_{a,b} \rangle$ ,  $\langle x^1, v_{a,b}, f^1 \rangle$  and  $\langle x^i \circ x^{i+1}, f^i, f^{i+1} \rangle$  are uniquely determined and isomorphic to  $A(1/2, c_3^1)$ ,  $A(1/2, c_4^1)$  and  $A(1/2, c_{i+4}^1)$  in (loc. cit.), respectively. Then by Theorem 4.6 of [LY16], we have the following relations in  $\text{Aut}(V)$ .

$$\tau_{u_{a,b}} = \tau_{v_{a,b}}, \quad \tau_{v_{a,b}} \tau_{f^1} = \tau_{x^1}, \quad \tau_{f^i} \tau_{f^{i+1}} = \begin{cases} \tau_{x^i \circ x^{i+1}} = \tau_{x^i} \tau_{x^{i+1}} & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

It is also shown in Remark 2.12 of [HLY12b] that  $\tau_{u_{a,b}} = \tau_{v_{a,b}}$  is trivial on  $V$ . Therefore,  $\tau_{f^1} = \tau_{x^1}$  and by induction we have

$$\tau_{f^n} = \begin{cases} \tau_{x^1} \tau_{x^2} \cdots \tau_{x^n} & \text{if } n \text{ is odd,} \\ \tau_{x^1} \tau_{x^2} \cdots \tau_{x^{n-1}} & \text{if } n > 0 \text{ is even.} \end{cases}$$

It is clear that  $\tau_{x^i}$  are trivial on the commutant  $\text{Com}_V X^{[n-1]}$  for  $1 \leq i \leq n-1$  since  $x^1, \dots, x^{n-1} \in X^{[n-1]}$ . Thus as an automorphism of  $\text{Com}_V X^{[n-1]}$  we have

$$\tau_{f^n} = \begin{cases} \varphi^{[n-1]}(\tau_{x^1} \cdots \tau_{x^n}) = \varphi^{[n-1]}(\tau_{x^n}) & \text{if } n \text{ is odd,} \\ \varphi^{[n-1]}(\tau_{x^1} \cdots \tau_{x^{n-1}}) = 1 & \text{if } n > 0 \text{ is even.} \end{cases}$$

If  $n > 0$  is even, it is shown in Theorem 4.8 of [LY16] that  $f^n$  is of  $\sigma$ -type on the commutant  $\text{Com}_V \langle f^{n-1} \rangle$  and both  $\sigma_{f^n}$  and  $\tau_{x^{n-1} \circ x^n}$  define the same automorphism on  $\text{Com}_V \langle f^{n-1} \rangle$ . Since  $\text{Com}_V X^{[n-1]} \subset \text{Com}_V \langle f^{n-1} \rangle$ , we have

$$\sigma_{f^n} = \varphi^{[n-1]}(\tau_{x^{n-1} \circ x^n}) = \varphi^{[n-1]}(\tau_{x^{n-1}} \tau_{x^n}) = \varphi^{[n-1]}(\tau_{x^n})$$

as an automorphism of  $\text{Com}_V X^{[n-1]}$ . This completes the proof. ■

## 5 Conway-Miyamoto correspondence

In this section, we apply our theorems to the moonshine vertex operator algebra and establish the Conway-Miyamoto correspondence for the Fischer 3-transposition groups.

Throughout this section, we identify the Monster simple group  $\mathbb{M}$  with the automorphism group of the moonshine VOA  $V^\natural$  (cf. [FLM88]). The moonshine VOA  $V^\natural$  can be defined over the real numbers and has a compact real form  $V_{\mathbb{R}}^\natural$  as in [FLM88, Mi04]. It is shown in Proposition 5.2 of [HLY12a] that all Ising vectors in  $V^\natural$  are contained in the compact real form  $V_{\mathbb{R}}^\natural$ . Therefore,  $V^\natural$  has a canonical real form defined as the real subalgebra generated by Ising vectors, which indeed coincides with  $V_{\mathbb{R}}^\natural$ . In the following discussion, we will consider subalgebras generated by Ising vectors of  $V^\natural$  so that all subalgebras have compact real forms.

**Definition 5.1.** Let  $V$  be a VOA and  $G$  a subgroup of  $\text{Aut}(V)$ . Let  $I$  be a conjugacy class of involutions of  $G$ . We define the *Conway-Miyamoto correspondence between involutions*  $I$  of  $G$  and  $c = c_n$  Virasoro vectors of  $V$  by means of the following conditions:

- (1) For each  $t \in I$ , there exists a unique  $c = c_n$  Virasoro vector  $e_t \in V^{C_G(t)}$ .
- (2) If the unique Virasoro vector  $e_t$  is not of  $\sigma$ -type on  $V$ , then  $\tau_{e_t} = t$  on  $V$ .
- (3) If  $e_t$  is of  $\sigma$ -type on  $V$  then  $\sigma_{e_t} = t$  on  $V$ .

The unique  $c = c_n$  Virasoro vector  $e_t$  of  $V^{C_G(t)}$  is called the *axial vector* associated to  $t$ . We say that the Conway-Miyamoto correspondence between  $G$  and  $V$  is *bijective* if the axial vector  $e_t$  is the unique  $c = c_n$  Virasoro vector  $a$  of  $V$  satisfying  $\tau_a = t$  (or  $\sigma_a = t$  in the case  $e_t$  is of  $\sigma$ -type on  $V$ ).

Here are known examples of Conway-Miyamoto correspondences.

**Theorem 5.2** ([ATLAS, FLM88, C85, Mi96, Ma01, H10, HLY12a, HLY12b]).

*There are Conway-Miyamoto correspondences between the following groups and VOAs.*

- (1) *2A-elements of the Monster and Ising vectors of  $V^\natural$ . The correspondence is bijective.*
- (2) *2A-elements of the Baby Monster and  $c = c_2$  Virasoro vectors of  $\sigma$ -type of  $VB^\natural = \text{Com}_{V^\natural}\langle e \rangle$ , where  $e$  is an Ising vector of  $V^\natural$ . The correspondence is bijective.*
- (3) *2C-elements of the largest Fischer 3-transposition group and  $c = c_4$  Virasoro vectors of  $\sigma$ -type of  $VF^\natural = \text{Com}_{V^\natural}\langle u_{a,b} \rangle$ , where  $a, b$  are Ising vectors of  $V^\natural$  such that  $(a | b) = 13 \cdot 2^{-10}$ .*

By the bijective correspondence in (1) of Theorem 5.2 and Proposition 2.5, we have

**Corollary 5.3.** *Let  $e^1, \dots, e^k$  be Ising vectors of  $V^\natural$ . Then the centralizer  $C_{\mathbb{M}}(\tau_{e^1}, \dots, \tau_{e^k})$  coincides with the pointwise stabilizer of  $\langle e^1, \dots, e^k \rangle$  in  $\mathbb{M}$ .*

Now we fix a pair of Ising vectors  $a, b$  in  $V^\natural$  such that  $\langle a, b \rangle$  is isomorphic to the 3A-algebra. Such a pair corresponds to a pair of 2A-involutions  $\tau_a$  and  $\tau_b$  such that  $\tau_a \tau_b$  is a 3A-element by (1) of Theorem 5.2, and is known to be unique up to conjugation by  $\mathbb{M}$  (cf. [ATLAS]).

Let  $E_{V^\natural}$  be the set of Ising vectors of  $V^\natural$  and  $I_{a,b} = \{x \in E_{V^\natural} \mid (a | x) = (b | x) = 2^{-5}\}$  as before. For  $n = 0, 1, 2$ , we consider the subalgebra  $X^{[n]} = \langle a, b, x^1, \dots, x^n \rangle$  of  $V^\natural$  defined as in Sec. 4.3 and study the automorphism group  $G^{[n]} = \langle \varphi^{[i]}(D^{[i]}) \rangle$  of  $\text{Com}_{V^\natural} X^{[n]}$  defined as in (4.18) and (4.19).

**Proposition 5.4.** *Let  $a, b, x^1, x^2 \in V^\natural$  be as above.*

- (1)  *$\langle D^{[0]} \rangle = C_{\mathbb{M}}(\tau_a, \tau_b) \cong \text{Fi}_{23}$  in  $\text{Aut}(V^\natural)$  and  $G^{[0]} \cong \text{Fi}_{23}$  in  $\text{Aut}(\text{Com}_{V^\natural} X^{[0]})$ . Moreover, for any  $\tau_y \in D^{[0]}$ ,  $\varphi^{[0]}(\tau_y)$  defines a 2A-element of  $\text{Fi}_{23}$  on  $\text{Aut}(\text{Com}_{V^\natural} X^{[0]})$ .*
- (2)  *$\langle D^{[1]} \rangle = C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}) \cong 2.\text{Fi}_{22}$  in  $\text{Aut}(V^\natural)$  and  $G^{[1]} \cong \text{Fi}_{22}$  in  $\text{Aut}(\text{Com}_{V^\natural} X^{[1]})$ . Moreover, for any  $\tau_y \in D^{[1]}$ ,  $\varphi^{[1]}(\tau_y)$  defines a 2A-element of  $\text{Fi}_{22}$  on  $\text{Aut}(\text{Com}_{V^\natural} X^{[1]})$ .*

(3)  $\langle D^{[2]} \rangle = C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}, \tau_{x^2}) \cong 2^2.\text{PSU}_6(2)$  and  $G^{[2]} \cong \text{PSU}_6(2)$  in  $\text{Aut}(\text{Com}_{V^\natural} X^{[2]})$ . Moreover, for any  $\tau_y \in D^{[2]}$ ,  $\varphi^{[2]}(\tau_y)$  defines a 2A-element of  $\text{PSU}_6(2)$  on  $\text{Aut}(\text{Com}_{V^\natural} X^{[2]})$ .

**Proof:** First, we observe that  $\langle D^{[n]} \rangle$  is a normal subgroup of  $C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}, \dots, \tau_{x^n})$  by Proposition 2.5.

(1): Since  $\langle D^{[0]} \rangle$  is a normal subgroup of  $C_{\mathbb{M}}(\tau_a, \tau_b)$  and  $C_{\mathbb{M}}(\tau_a, \tau_b) \cong \text{Fi}_{23}$  is a simple group by [ATLAS], we have  $\langle D^{[0]} \rangle = C_{\mathbb{M}}(\tau_a, \tau_b) \cong \text{Fi}_{23}$ . By definition,  $G^{[0]}$  is a homomorphic image of  $\langle D^{[0]} \rangle$  and we have  $G^{[0]} \cong \text{Fi}_{23}$  in  $\text{Aut}(\text{Com}_{V^\natural} X^{[0]})$ . By Corollary 3.7,  $D^{[0]}$  is the set of 3-transpositions, and only the 2A-involutions satisfy the 3-transposition property in  $\text{Fi}_{23}$  (cf. [ATLAS, I76]). Therefore, each involution in  $D^{[0]}$  defines a 2A-element of  $\text{Fi}_{23}$ .

(2): By (1) and Corollary 5.3 we have  $C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}) \cong C_{\text{Fi}_{23}}(2A) \cong 2.\text{Fi}_{22}$ . Since  $\langle D^{[1]} \rangle$  is normal in  $C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1})$ , we have  $\langle D^{[1]} \rangle = C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}) \cong 2.\text{Fi}_{22}$  in  $\text{Aut}(V^\natural)$  where the center is generated by  $\tau_{x^1}$ . Then  $G^{[1]} \cong \text{Fi}_{22}$  since the center  $\tau_{x^1}$  acts trivially on  $\text{Aut}(\text{Com}_{V^\natural} X^{[1]})$  by Lemma 4.19. Again it follows from Corollary 3.7 that  $\langle \varphi^{[1]}(D^{[1]}) \rangle$  is the set of 3-transpositions in  $G^{[1]} \cong \text{Fi}_{22}$  and only 2A-involutions satisfy the 3-transposition property in  $\text{Fi}_{22}$  (cf. [ATLAS, I76]). Therefore  $\varphi^{[1]}(\tau_y)$  defines a 2A-element of  $\text{Fi}_{22}$  on  $\text{Com}_{V^\natural} X^{[1]}$  for any  $\tau_y \in D^{[1]}$ .

(3): By (2) and Corollary 5.3, we have  $\langle D^{[2]} \rangle = C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}, \tau_{x^2}) \cong 2^2.\text{PSU}_6(2)$ . Since the center  $\langle \tau_{x^1}, \tau_{x^2} \rangle$  acts trivially on  $\text{Com}_{V^\natural} X^{[2]}$ , we have  $G^{[2]} \cong \text{PSU}_6(2)$  in  $\text{Aut}(\text{Com}_{V^\natural} X^{[2]})$ . Since only 2A-involutions satisfy the 3-transposition property in  $\text{PSU}_6(2)$ ,  $\varphi^{[2]}(\tau_y)$  is a 2A-element for any  $\tau_y \in D^{[2]}$ . This completes the proof. ■

**Lemma 5.5.** *Let  $\omega^0, \omega^1$  and  $\omega^2$  be the conformal vectors of the subalgebras  $X^{[0]} = \langle a, b \rangle$ ,  $X^{[1]} = \langle a, b, x^1 \rangle$  and  $X^{[2]} = \langle a, b, x^1, x^2 \rangle$  of  $V^\natural$ , respectively, and let  $f^1 = \omega^1 - \omega^0$  and  $f^2 = \omega^2 - \omega^1$  be the conformal vectors of  $\text{Com}_{X^{[1]}} X^{[0]}$  and  $\text{Com}_{X^{[2]}} X^{[1]}$ , respectively. Then  $f^1$  is fixed by  $C_{G^{[0]}}(\varphi^{[0]}(\tau_{x^1}))$  and  $f^2$  is fixed by  $C_{G^{[1]}}(\varphi^{[1]}(\tau_{x^2}))$ .*

**Proof:** It follows from Proposition 5.4 that  $C_{G^{[0]}}(\varphi^{[0]}(\tau_{x^1})) = \varphi^{[0]}(C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}))$  and  $C_{G^{[1]}}(\varphi^{[1]}(\tau_{x^2})) = \varphi^{[1]}(C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}, \tau_{x^2}))$ . Then  $C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1})$  fixes  $f^1 = \omega^1 - \omega^0 \in \langle a, b, x^1 \rangle$  and  $C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}, \tau_{x^2})$  fixes  $f^2 = \omega^2 - \omega^1 \in \langle a, b, x^1, x^2 \rangle$  by Corollary 5.3. Thus the claim follows. ■

## 5.1 Transpositions of $\text{Fi}_{23}$ and $c = 25/28$ Virasoro vectors

Let  $a, b$  be Ising vectors of  $V^\natural$  such that  $\langle a, b \rangle$  is the 3A-algebra. In this subsection, we consider  $X^{[0]} = \langle a, b \rangle \subset V^\natural$  and its commutant subalgebra  $\text{Com}_{V^\natural} \langle a, b \rangle$ . Let

$$D^{[0]} = \{\tau_x \in \text{Aut}(V^\natural) \mid x \in I_{a,b}\}, \quad G^{[0]} = \langle \varphi^{[0]}(D^{[0]}) \rangle \subset \text{Aut}(\text{Com}_{V^\natural} \langle a, b \rangle)$$

be defined as in (4.18) and (4.19). We have shown in Proposition 5.4 that  $G^{[0]} \cong \text{Fi}_{23}$ . In the following, we will identify  $\text{Fi}_{23}$  with  $G^{[0]} \subset \text{Aut}(\text{Com}_{V^\natural} \langle a, b \rangle)$ . Let  $x^1 \in D^{[0]}$

and  $X^{[1]} = \langle a, b, x^1 \rangle$ . Let  $f^1$  be the conformal vector of  $\text{Com}_{X^{[1]}} X^{[0]}$ . Then it follows from Theorem 4.20 and Proposition 5.4 that  $\tau_{f^1} = \varphi^{[0]}(\tau_{x^1})$  defines a 2A-element of  $\text{Fi}_{23}$ . Moreover, it is shown in Lemma 5.5 that  $f^1$  is fixed by the centralizer  $C_{\text{Fi}_{23}}(\tau_{f^1})$ . We will show that  $f^1$  is the unique  $C_{\text{Fi}_{23}}(\tau_{f^1})$ -invariant  $c = c_5$  Virasoro vector of  $\text{Com}_{V^\natural} \langle a, b \rangle$ .

Recall that the 3A-algebra  $U_{3A} = \langle a, b \rangle$  has 6 irreducible representations  $U(h)$  which are distinguished by the top weights  $h \in \{0, 1/7, 5/7, 2/5, 19/35, 4/35\}$  (cf. [SY03, LLY03]). We denote by  $\text{Irr } U_{3A}$  the set of equivalent classes of irreducible  $U_{3A}$ -modules. For  $U(h) \in \text{Irr } U_{3A}$ , we set

$$U(h)^c := \text{Hom}_{\langle a, b \rangle}(U(h), V^\natural).$$

Then we have the isotypical decomposition

$$V^\natural = \bigoplus_{U(h) \in \text{Irr } \langle a, b \rangle} U(h) \otimes U(h)^c. \quad (5.1)$$

The commutant  $\text{Com}_{V^\natural} \langle a, b \rangle$  acts naturally on  $U(h)^c$  and one can consider (5.1) as the decomposition as a  $\langle a, b \rangle \otimes \text{Com}_{V^\natural} \langle a, b \rangle$ -module. We denote the top levels of  $U(h)$  and  $U(h)^c$  by  $\text{Top } U(h)$  and  $\text{Top } U(h)^c$ , respectively. The top weight and the dimension of the top level of  $U(h)^c$  are obtained in [HLY12b].

**Lemma 5.6.** *The top levels of  $U(h)^c$  are irreducible as  $G^{[0]} \cong \text{Fi}_{23}$ -modules and their characters are as in the following table.*

$U(h)$	$U(0)$	$U(5/7)$	$U(19/35)$	$U(2/5)$	$U(1/7)$	$U(4/35)$
$\dim \text{Top } U(h)$	1	3	3	1	1	2
Top weight of $U(h)^c$	0	9/7	51/35	8/5	13/7	66/35
$\dim \text{Top } U(h)^c$	1	782	3588	5083	25806	60996
$\text{Fi}_{23}$ -character	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_7$

Here  $\chi_i$  denotes the irreducible characters of  $\text{Fi}_{23}$  labeled as in [ATLAS, page 178]. By abuse of notations, we also use  $\chi_i$  to denote the irreducible module affording the character  $\chi_i$ . Then we have the decomposition  $(\text{Com}_{V^\natural} \langle a, b \rangle)_2 = \chi_1 + \chi_6 (= \mathbf{1} + \mathbf{30888})$  as a  $\text{Fi}_{23}$ -module.

The centralizer  $C_{\text{Fi}_{23}}(\tau_{f^1})$  is isomorphic to  $2.\text{Fi}_{22}$  by Proposition 5.4 and we have the following decomposition (cf. [ATLAS]).

**Proposition 5.7.** *Consider the action of the centralizer  $C_{G^{[0]}}(\varphi^{[0]}(\tau_{x^1})) \cong 2.\text{Fi}_{22}$  on the Griess algebra of  $\text{Com}_{V^\natural} \langle a, b \rangle$  and the top levels  $\text{Top } U(h)^c$  of  $U(h)^c$  in (5.1). As  $2.\text{Fi}_{22}$ -*

modules, we have the following decompositions.

$$(\text{Com}_{V^{\mathfrak{h}}} \langle a, b \rangle)_2 = \frac{\mathbf{1}}{\chi_1} + \frac{\mathbf{1}}{\chi_1} + \frac{\mathbf{429}}{\chi_3} + \frac{\mathbf{3080}}{\chi_7} + \frac{\mathbf{13650}}{\chi_9} + \frac{\mathbf{13728}}{\chi_{73}},$$

$$\text{Top } U(5/7)^c = \frac{\mathbf{1}}{\chi_1} + \frac{\mathbf{429}}{\chi_3} + \frac{\mathbf{352}}{\chi_{66}},$$

$$\text{Top } U(19/35)^c = \frac{\mathbf{78}}{\chi_2} + \frac{\mathbf{1430}}{\chi_5} + \frac{\mathbf{2080}}{\chi_{68}},$$

$$\text{Top } U(2/5)^c = \frac{\mathbf{3003}}{\chi_6} + \frac{\mathbf{2080}}{\chi_{67}},$$

$$\text{Top } U(1/7)^c = \frac{\mathbf{1001}}{\chi_4} + \frac{\mathbf{10725}}{\chi_8} + \frac{\mathbf{352}}{\chi_{66}} + \frac{\mathbf{13728}}{\chi_{74}},$$

$$\text{Top } U(4/35)^c = \frac{\mathbf{1430}}{\chi_5} + \frac{\mathbf{30030}}{\chi_{10}} + \frac{\mathbf{2080}}{\chi_{67}} + \frac{\mathbf{27456}}{\chi_{75}},$$

where  $\chi_i$  are the irreducible characters of  $\text{Fi}_{22}$  and  $2.\text{Fi}_{22}$  labeled as in [ATLAS, pages 156–157].

By Lemma 5.5 and Proposition 5.7, the Griess algebra of the  $C_{\text{Fi}_{23}}(\tau_{f^1})$ -invariants of  $\text{Com}_{V^{\mathfrak{h}}} \langle a, b \rangle$  is 2-dimensional spanned by the  $c = 782/35$  conformal vector and the  $c = 25/28$  Virasoro vector  $f^1 \in \text{Com}_{X^{[1]}} X^{[0]}$ . Therefore,  $f^1$  is the axial vector of the 2A-element  $\tau_{f^1}$  of  $\text{Fi}_{23}$  and we have established the Conway-Miyamoto correspondence for  $\text{Fi}_{23}$ .

**Theorem 5.8.** *Let  $E_{V^{\mathfrak{h}}}$  be the set of Ising vectors of  $V^{\mathfrak{h}}$  and let  $a, b \in E_{V^{\mathfrak{h}}}$  be a pair such that  $(a | b) = 13 \cdot 2^{-10}$ . Then there exists a Conway-Miyamoto correspondence between 2A-elements of  $\text{Fi}_{23}$  and  $c = 25/28$  Virasoro vectors of  $\text{Com}_{V^{\mathfrak{h}}} \langle a, b \rangle$ . More precisely, set  $I_{a,b} = \{x \in E_{V^{\mathfrak{h}}} \mid (a | x) = (b | x) = 2^{-5}\}$ . Then the following hold.*

- (1)  $C_{\mathbb{M}}(\tau_a, \tau_b) = \langle \tau_x \mid x \in I_{a,b} \rangle \cong \text{Fi}_{23}$  acts faithfully on  $\text{Com}_{V^{\mathfrak{h}}} \langle a, b \rangle$ .
- (2) There is a one to one correspondence between  $I_{a,b}$  and 2A-involutions of  $\text{Fi}_{23}$  via the Miyamoto involution  $x \mapsto \tau_x|_{\text{Com}_{V^{\mathfrak{h}}} \langle a, b \rangle}$ .
- (3) Given a 2A-involution  $t$  of  $\text{Fi}_{23}$ , there exists a unique  $C_{\text{Fi}_{23}}(t)$ -invariant  $c = 25/28$  Virasoro vector  $f$  of  $\text{Com}_{V^{\mathfrak{h}}} \langle a, b \rangle$  such that  $\tau_f = t$ . Moreover,  $\tau_f$  on  $V^{\mathfrak{h}}$  defines a 2A-involution of  $\mathbb{M}$ .

For the inductive argument from  $G^{[0]} = \text{Fi}_{23}$  to  $G^{[1]} = \text{Fi}_{22}$ , we determine the decomposition of the Griess algebra of  $\text{Com}_{V^{\mathfrak{h}}} X^{[1]}$  as a  $\text{Fi}_{22}$ -module.

**Lemma 5.9.** *Consider the subalgebras  $X^{[0]} = \langle a, b \rangle \subset X^{[1]} = \langle a, b, x^1 \rangle \subset V^{\mathfrak{h}}$  as above and let  $f^1$  be the conformal vector of  $\text{Com}_{X^{[1]}} X^{[0]}$ . Then the zero-mode  $o(f^1)$  acts on the Griess algebra of  $V^{\mathfrak{h}}$  semisimply with possible eigenvalues*

$$0, 2, \frac{9}{7}, \frac{3}{4}, \frac{5}{14}, \frac{1}{28}, \frac{3}{28}, \frac{15}{28}, \frac{5}{32}, \frac{3}{224}, \frac{15}{224}, \frac{99}{224}, \frac{143}{224}.$$

More precisely,  $\mathfrak{o}(f^1)$  acts semisimply on the Griess algebra of  $\text{Com}_{V^\natural}\langle a, b \rangle$  and the top levels  $\text{Top } U(h)^c$  with possible eigenvalues as follows.

$$\begin{aligned} (\text{Com}_{V^\natural}\langle a, b \rangle)_2 &: 0, 3/4, 5/32, & \text{Top } U(5/7)^c &: 9/7, 1/28, 15/224, \\ \text{Top } U(2/5)^c &: 0, 5/32, & \text{Top } U(1/7)^c &: 5/14, 3/28, 143/224, 3/224, \\ \text{Top } U(19/35)^c &: 5/14, 3/28, 3/224, & \text{Top } U(4/35)^c &: 15/28, 1/28, 99/224, 15/224. \end{aligned}$$

**Proof:** Since the axial vector  $f^1$  is fixed by  $C_{\text{Fi}_{23}}(\tau_{f^1})$  by Theorem 5.8, its zero-mode  $\mathfrak{o}(f^1)$  acts as a scalar on each  $C_{\text{Fi}_{23}}(\tau_{f^1})$ -irreducible component. By (5.1), we have the following decomposition of the Griess algebra of  $V^\natural$ .

$$V_2^\natural = \langle a, b \rangle_2 \oplus (\text{Com}_{V^\natural}\langle a, b \rangle)_2 \oplus \bigoplus_{\substack{U(h) \in \text{Irr } \langle a, b \rangle \\ h > 0}} \text{Top } U(h) \otimes \text{Top } U(h)^c. \quad (5.2)$$

The 6A-algebra  $\langle a, b \circ x^1 \rangle = \langle a, b, x^1 \rangle$  contains a rational full sub VOA  $\langle a, b, f^1 \rangle = \langle a, b \rangle \otimes \langle f^1 \rangle$  by (8) of Theorem 2.13. As explained in the beginning of this section, all Ising vectors of  $V^\natural$  are contained in the compact real form  $V_{\mathbb{R}}^\natural$  of  $V^\natural$  so that  $V^\natural$  is semisimple as a  $\langle a, b, x^1 \rangle$ -module. The classification of irreducible modules over the 6A-algebra is not completed so far but we can determine their possible structures as  $\langle a, b \rangle \otimes \langle f^1 \rangle$ -modules by the fusion rules of  $U_{3A}$ -modules (cf. [SY03]) and  $L(25/28, 0)$ -modules. An irreducible  $U_{6A}$ -module is isomorphic to one of the following as a  $U_{3A} \otimes L(25/28, 0)$ -module.

$$\begin{aligned} [0, 0] \oplus [1/7, 34/7] \oplus [5/7, 9/7], & \quad [0, 5/32] \oplus [1/7, 675/224] \oplus [5/7 \otimes 99/224], \\ [1/7, 3/224] \oplus [0, 165/32] \oplus [5/7, 323/224], & \quad [19/35, 3/224] \oplus [2/5, 165/32] \oplus [4/35, 323/224], \\ [1/7, 5/14] \oplus [0, 15/2] \oplus [5/7, 39/14], & \quad [19/35, 5/14] \oplus [2/5, 15/2] \oplus [4/35, 39/14], \\ [2/5, 0] \oplus [19/35, 34/7] \oplus [4/35, 9/7], & \quad [4/35, 15/224] \oplus [2/5, 57/32] \oplus [19/35, 143/224], \\ [4/35, 1/28] \oplus [2/5, 3/4] \oplus [19/35, 45/28], & \quad [1/7, 3/28] \oplus [0, 13/4] \oplus [5/7, 15/28], \\ [0, 3/4] \oplus [5/7, 1/28] \oplus [1/7, 45/28], & \quad [1/7, 143/224] \oplus [5/7, 15/224] \oplus [0, 57/32], \\ [19/35, 3/28] \oplus [4/35, 15/28] \oplus [2/5, 13/4], & \quad [2/5, 5/32] \oplus [4/35, 99/224] \oplus [19/35, 675/224], \end{aligned} \quad (5.3)$$

where  $[h, k]$  denotes  $U(h) \otimes L(25/28, k)^{\oplus m_k}$  with some multiplicities  $m_k$ , which are expected to be 1 in all cases. By considering the decomposition in (5.2) together with the possible shapes in (5.3), we obtain the list of possible eigenvalues of  $\mathfrak{o}(f^1)$  as in the assertion. ■

**Remark 5.10.** The Griess algebras of dihedral subalgebras are classified in [S07] (cf. Theorem 2.6) but the uniqueness of the VOA structure of the 6A-algebra has not been established so far. However, the pair of Ising vectors of  $V^\natural$  generating a 6A-algebra is unique up to conjugation by  $\mathbb{M}$  (cf. (1) of Theorem 5.2 and [ATLAS]), and therefore inside  $V^\natural$  the structure of a 6A-subalgebra is uniquely determined (cf. [LYY05, LM06]). Indeed, one can

determine the multiplicities  $m_k$  by using the so-called  $W_k$ -algebras  $W_k(2(k-1)/(k+2))$  related to parafermion algebras (cf. [DLY09, LYY05]). Namely,  $U_{6A}$  is a simple current extension of  $W_2(1/2) \otimes W_3(4/5) \otimes W_6(5/4)$  where we know  $W_2(1/2) \cong L(1/2, 0)$  and  $W_3(4/5) \cong L(4/5, 0) \oplus L(4/5, 3)$ . The fusion rules of  $W_6(5/4)$ -modules is isomorphic to that of the level 2 affine VOA of type  $A_5^{(1)}$  (cf. [DW16]) and it is possible to determine the precise multiplicities  $m_k$  above by using fusion rules and the vacuum characters of irreducible  $W_6(5/4)$ -modules as in Appendix B.3 of [LYY05]. However, at this moment we only need possible eigenvalues of  $o(f^1)$  and we do not include them in this paper.

In order to determine the  $o(f^1)$ -spectrum of the Griess algebra of  $V^\natural$ , we need traces  $\text{tr}_{V_2^\natural} o(f)^i$  for  $1 \leq i \leq 5$ , which can be computed by the Matsuo-Norton trace formulae in [Ma01].

**Lemma 5.11** ([Ma01]). *The traces of  $o(f^1)^i$ ,  $1 \leq i \leq 5$ , on the Griess algebra are as follows.*

$$\begin{aligned} \text{tr}_{V_2^\natural} o(f^1) &= \frac{410175}{28}, & \text{tr}_{V_2^\natural} o(f^1)^2 &= \frac{2411375}{784}, & \text{tr}_{V_2^\natural} o(f^1)^3 &= \frac{27230625}{21952}, \\ \text{tr}_{V_2^\natural} o(f^1)^4 &= \frac{793401325}{1229312}, & \text{tr}_{V_2^\natural} o(f^1)^5 &= \frac{15221783625}{39337984}. \end{aligned}$$

By Lemma 5.11 and possible eigenvalues in Lemma 5.9, we can determine the  $o(f^1)$ -spectrum compatible with the decompositions in Proposition 5.7. The result is as follows.



**Proposition 5.12.** *The zero-mode  $o(f^1)$  acts on the decomposition in (5.2) as follows.*

$$\begin{array}{rcll} (\text{Com}_{V^\natural} \langle a, b \rangle)_2 & = & \mathbf{1} & + \mathbf{1} + \mathbf{429} + \mathbf{3080} + \mathbf{13650} + \mathbf{13728} \\ o(f^1) & : & 2 & 0 \quad 3/4 \quad 0 \quad 0 \quad 5/32 \\ \tau_{f^1} & : & 1 & 1 \quad 1 \quad 1 \quad 1 \quad -1 \end{array}$$

$$\begin{array}{rcll} \text{Top } U(5/7)^c & = & \mathbf{1} & + \mathbf{429} + \mathbf{352} \\ o(f^1) & : & 9/7 & 1/28 \quad 15/224 \\ \tau_{f^1} & : & 1 & 1 \quad -1 \end{array}$$

$$\begin{array}{rcll} \text{Top } U(19/35)^c & = & \mathbf{78} & + \mathbf{1430} + \mathbf{2080} \\ o(f^1) & : & 5/14 & 3/28 \quad 3/224 \\ \tau_{f^1} & : & 1 & 1 \quad -1 \end{array}$$

$$\begin{array}{rcll} \text{Top } U(2/5)^c & = & \mathbf{3003} & + \mathbf{2080} \\ o(f^1) & : & 0 & 5/32 \\ \tau_{f^1} & : & 1 & -1 \end{array}$$

$$\begin{array}{rcll} \text{Top } U(1/7)^c & = & \mathbf{1001} & + \mathbf{10725} + \mathbf{352} + \mathbf{13728} \\ o(f^1) & : & 5/14 & 3/28 \quad 143/224 \quad 3/224 \\ \tau_{f^1} & : & 1 & 1 \quad -1 \quad -1 \end{array}$$

$$\begin{array}{rcll} \text{Top } U(4/35)^c & = & \mathbf{1430} & + \mathbf{30030} + \mathbf{2080} + \mathbf{27456} \\ o(f^1) & : & 15/28 & 1/28 \quad 99/224 \quad 15/224 \\ \tau_{f^1} & : & 1 & 1 \quad -1 \quad -1 \end{array}$$

**Proof:** Since  $\tau_{f^1}$  is the central element in  $C_{\text{Fi}_{23}}(\tau_{f^1}) \cong 2.\text{Fi}_{22}$ , it acts as 1 on the irreducible  $2.\text{Fi}_{22}$ -components corresponding to the characters  $\chi_i$  with  $1 \leq i \leq 10$  and as  $-1$  on those to  $\chi_i$  with  $i \geq 66$  (cf. [ATLAS]). If  $\tau_{f^1}$  is trivial then the possible eigenvalues of  $o(f^1)$  are 0, 2,  $9/7$ ,  $3/4$ ,  $5/14$ ,  $1/28$ ,  $3/28$ ,  $15/28$  and if  $\tau_{f^1}$  is non-trivial then the possible eigenvalues are  $5/32$ ,  $3/224$ ,  $15/224$ ,  $99/224$ ,  $143/224$  by Lemma 5.9. This information together with Lemma 5.9 leads to the unique assignment of eigenvalues compatible with the set of traces in Lemma 5.11 and we obtain the eigenspace decompositions as in the assertion.  $\blacksquare$

## 5.2 Transpositions of $\text{Fi}_{22}$ and $c = 11/12$ Virasoro vectors

Let  $X^{[0]} = \langle a, b \rangle \subset X^{[1]} = \langle a, b, x^1 \rangle \subset X^{[2]} = \langle a, b, x^1, x^2 \rangle \subset V^\natural$  be the subalgebras of  $V^\natural$  defined as in (4.12), and let  $f^1$  and  $f^2$  be the conformal vectors of  $\text{Com}_{X^{[1]}} X^{[0]}$  and  $\text{Com}_{X^{[2]}} X^{[1]}$ , respectively. Let

$$D^{[1]} = \{\tau_y \in D^{[0]} \mid \tau_y \tau_{x^1} = \tau_{x^1} \tau_y\}, \quad G^{[1]} = \langle \varphi^{[1]}(D^{[1]}) \rangle \subset \text{Aut}(\text{Com}_{V^\natural} \langle a, b, x^1 \rangle)$$

be defined as in (4.18) and (4.19). We have shown in Proposition 5.4 that  $G^{[1]} \cong \text{Fi}_{22}$ . In the following we will identify  $\text{Fi}_{22}$  with  $G^{[1]}$ . Since  $\langle a, b, f^1 \rangle \cong \langle a, b \rangle \otimes \langle f^1 \rangle$  is a full subalgebra of  $X^{[1]} = \langle a, b, x^1 \rangle$ , the commutant  $\text{Com}_{V^\natural} \langle a, b, x^1 \rangle$  coincides with the 0-eigenspace of

$o(f^1)$  in  $\text{Com}_{V^\natural}\langle a, b \rangle$ . Therefore, by Proposition 5.12, we have the following decomposition of the Griess algebra of  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$  as a  $\text{Fi}_{22}$ -module.

**Proposition 5.13.** *As a  $G^{[1]} \simeq \text{Fi}_{22}$ -module, the Griess algebra of  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$  decomposes as follows.*

$$(\text{Com}_{V^\natural}\langle a, b, x^1 \rangle)_2 = \underset{\chi_1}{\mathbf{1}} + \underset{\chi_7}{\mathbf{3080}} + \underset{\chi_9}{\mathbf{13650}},$$

where  $\chi_1$ ,  $\chi_7$  and  $\chi_9$  are irreducible characters of  $\text{Fi}_{22}$  labeled as in [ATLAS, page 156].

It is shown in Proposition 5.4 that  $\varphi^{[1]}(\tau_{x^2})$  defines a 2A-element of  $\text{Fi}_{22}$  and its centralizer is  $C_{\text{Fi}_{22}}(\varphi^{[1]}(\tau_{x^2})) \cong 2.\text{PSU}_6(2)$ . We have the following decompositions.

**Lemma 5.14.** *As a  $C_{\text{Fi}_{22}}(\varphi^{[1]}(\tau_{x^2})) \cong 2.\text{PSU}_6(2)$ -module, we have*

$$\begin{aligned} \underset{\chi_7}{\mathbf{3080}} &= \underset{\mu_1}{\mathbf{1}} + \underset{\mu_4}{\mathbf{252}} + \underset{\mu_6}{\mathbf{440}} + \underset{\mu_{11}}{\mathbf{1155}} + \underset{\mu_{51}}{\mathbf{1232}}, \\ \underset{\chi_9}{\mathbf{13650}} &= \underset{\mu_4}{\mathbf{252}} + \underset{\mu_{12}}{\mathbf{1155}} + \underset{\mu_{13}}{\mathbf{1155}} + \underset{\mu_{20}}{\mathbf{4928}} + \underset{\mu_{57}}{\mathbf{6160}}, \end{aligned}$$

where  $\chi_i$  are the irreducible characters of  $\text{Fi}_{22}$  as in the previous proposition, and  $\mu_i$  are the irreducible characters of  $\text{PSU}_6(2)$  and  $2.\text{PSU}_6(2)$  labeled as in [ATLAS, page 116].

It follows from Lemma 5.5, Proposition 5.13 and Lemma 5.14 that the Griess algebra of the  $C_{\text{Fi}_{22}}(\varphi^{[1]}(\tau_{x^2}))$ -invariants of  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$  is 2-dimensional spanned by the  $c = 429/20$  conformal vector and the  $c = 11/12$  Virasoro vector  $f^2$  in  $\text{Com}_{X^{[2]}}X^{[1]}$ . It also follows from (2) of Theorem 4.20 that  $f^2$  is of  $\sigma$ -type and satisfies  $\sigma_{f^2} = \varphi^{[1]}(\tau_{x^2})$  on  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$ . Therefore, we have obtained the Conway-Miyamoto correspondence between 2A-involutions of  $\text{Fi}_{22}$  and  $c = 11/12$  Virasoro vectors of  $\sigma$ -type of  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$ .

**Theorem 5.15.** *Let  $E_{V^\natural}$  be the set of Ising vectors of  $V^\natural$  and let  $a, b \in E_{V^\natural}$  be a pair such that  $(a|b) = 13 \cdot 2^{-10}$ . Set  $I_{a,b} = \{x \in E_{V^\natural} \mid (a|x) = (b|x) = 2^{-5}\}$  and take  $x^1 \in I_{a,b}$ . Then there exists a Conway-Miyamoto correspondence between 2A-elements of  $\text{Fi}_{22}$  and  $c = 11/12$  Virasoro vectors of  $\sigma$ -type of  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$ . More precisely, the following hold.*

- (1) *Set  $D^{[1]} = \{\tau_y \mid y \in I_{a,b}, \tau_{x^1}\tau_y = \tau_y\tau_{x^1}\}$ . Then  $\langle D^{[1]} \rangle = C_{\mathbb{M}}(\tau_a, \tau_b, \tau_{x^1}) \cong 2.\text{Fi}_{22}$  acts on  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$  with the kernel  $\langle \tau_{x^1} \rangle$ .*
- (2) *Let  $\varphi^{[1]} : \langle D^{[1]} \rangle \rightarrow \text{Aut}(\text{Com}_{V^\natural}\langle a, b, x^1 \rangle)$  be the homomorphism given in (1) and  $G^{[1]}$  the image of  $\varphi^{[1]}$ . Then  $G^{[1]} \cong \text{Fi}_{22}$  and  $\varphi^{[1]}(D^{[1]})$  is the set of 2A-involutions of  $\text{Fi}_{22}$ .*
- (3) *Given a 2A-involution  $t$  of  $\text{Fi}_{22}$ , there exists a unique  $C_{\text{Fi}_{22}}(t)$ -invariant  $c = 11/12$  Virasoro vector  $f$  of  $\sigma$ -type of  $\text{Com}_{V^\natural}\langle a, b, x^1 \rangle$  such that  $\sigma_f = t$ .*

**Remark 5.16.** In principle, we can continue to consider the next case  $X^{[2]} \subset X^{[3]} = \langle a, b, x^1, x^2, x^3 \rangle$  and study  $\text{Com}_{V^\natural} X^{[2]}$  as a  $G^{[2]} \cong \text{PSU}_6(2)$ -module. Let  $f^3$  be the  $c = c_7 = 14/15$  Virasoro vector of  $\text{Com}_{X^{[3]}} X^{[2]}$ . We know that  $\tau_{f^3} = \varphi^{[2]}(\tau_{x^3})$  defines a 2A-element of  $\text{PSU}_2(6)$  on  $\text{Com}_{V^\natural} X^{[2]}$  by (1) of Theorem 4.20 and Proposition 5.4. In order to verify the Conway-Miyamoto correspondence for  $\text{Fi}_{21} = \text{PSU}_6(2)$ , we have to compute the  $C_{\text{Fi}_{21}}(\tau_{f^3})$ -invariants of the Griess algebra of  $\text{Com}_{V^\natural} X^{[2]}$ . However, the Griess algebra of  $\text{Com}_{V^\natural} X^{[1]}$  splits into many pieces as in Lemma 5.14 and its 0-eigenspace of  $\mathfrak{o}(f^2)$  is technically difficult to determine so that we cannot obtain the decomposition of the Griess algebra of  $\text{Com}_{V^\natural} X^{[2]}$  at this moment. It is likely that the  $C_{\text{Fi}_{21}}(\tau_{f^3})$ -invariants of the Griess algebra of  $\text{Com}_{V^\natural} X^{[2]}$  has a dimension more than 2, and the Conway-Miyamoto correspondence for  $\text{Fi}_{21}$  seems to fail. However, even the  $C_{\text{Fi}_{21}}(\tau_{f^3})$ -invariants is larger, we still have a chance that it contains a unique  $c = c_7$  Virasoro vector.

## Appendix A Explicit constructions

We will give explicit constructions of the subalgebras discussed in Section 4.

### A.1 Compact real form of a lattice VOA

Let  $L$  be an even positive definite lattice. Let  $V_L$  be the associated lattice VOA defined over  $\mathbb{C}$  and  $V_{L,\mathbb{R}}$  the one defined over  $\mathbb{R}$  (cf. [FLM88]). Let  $\theta$  be a lift of the  $(-1)$ -isometry of  $L$  on  $V_{L,\mathbb{R}}$  and  $V_{L,\mathbb{R}} = V_{L,\mathbb{R}}^+ \oplus V_{L,\mathbb{R}}^-$  the eigenspace decomposition such that  $\theta$  acts by  $\pm 1$  on  $V_{L,\mathbb{R}}^\pm$ . Then the invariant bilinear form on  $V_{L,\mathbb{R}}$  is positive definite on  $V_{L,\mathbb{R}}^+$  and negative definite on  $V_{L,\mathbb{R}}^-$ . Therefore, the real subspace

$$(V_L)_{\mathbb{R}} := V_{L,\mathbb{R}}^+ \oplus \sqrt{-1}V_{L,\mathbb{R}}^- \quad (\text{A.1})$$

forms a compact real subalgebra of  $V_L$ . In the following We will use the compact form  $(V_L)_{\mathbb{R}}$  in (A.1) of  $V_L$ .

**Remark A.1.** In the construction of a lattice vertex operator algebra  $V_L$  one needs to implement a 2-cocycle  $\varepsilon \in Z^2(L, \{\pm 1\})$  such that  $\varepsilon(a, b)\varepsilon(b, a) = (-1)^{(a|b)}$  for  $a, b \in L$ . If  $L$  is of rank one or doubly even, then we can take  $\varepsilon$  to be trivial.

### A.2 Virasoro vectors associated to root systems

We recall the construction of certain Virasoro vectors from [DLMN98]. Let  $R$  be a root lattice and  $\Phi(R)$  its root system. Fix a system of simple roots and let  $\Phi^+(R)$  and  $\Phi^-(R)$  be the set of positive and negative roots, respectively. We use  $\sqrt{2}R$  to denote the  $\mathbb{Z}$ -submodule  $\sqrt{2} \otimes_{\mathbb{Z}} R$  of  $\mathbb{R} \otimes_{\mathbb{Z}} R$  which forms a doubly even lattice. By [DLMN98], the

conformal vector  $\omega_R$  of the lattice VOA  $V_{\sqrt{2}R}$  is given by

$$\omega_R = \frac{1}{4h} \sum_{\alpha \in \Phi(R)} \alpha_{(-1)}^2 \mathbb{1},$$

where  $h$  is the Coxeter number of  $R$ . For  $\alpha \in \sqrt{2}R$  with  $(\alpha|\alpha) = 4$  we set

$$w^\pm(\alpha) := \frac{1}{16} \alpha_{(-1)}^2 \mathbb{1} \pm \frac{1}{4} (e^\alpha + e^{-\alpha}). \quad (\text{A.2})$$

Then  $w^\pm(\alpha)$  are Ising vectors (cf. [DMZ94]). The following is straight forward.

**Lemma A.2.** *Let  $\alpha, \beta \in \sqrt{2}R$  be squared norm 4 vectors. Then*

$$(w^\varepsilon(\alpha) | w^{\varepsilon'}(\beta)) = \begin{cases} 2^{-2} & \text{if } \alpha = \pm\beta \text{ and } \varepsilon = \varepsilon', \\ 0 & \text{if } (\alpha|\beta) = 0 \text{ or } \alpha = \pm\beta \text{ with } \varepsilon = -\varepsilon', \\ 2^{-5} & \text{if } (\alpha|\beta) = \pm 2, \end{cases}$$

where  $w^\pm(\alpha)$  and  $w^\pm(\beta)$  are defined as in (A.2). Moreover, we have

$$w^\varepsilon(\alpha)_{(1)} w^{\varepsilon'}(\beta) = \frac{1}{4} \left( w^\varepsilon(\alpha) + w^{\varepsilon'}(\beta) - w^{-\varepsilon\varepsilon'}(\alpha - \beta) \right)$$

if  $(\alpha|\beta) = \pm 2$ .

We also define

$$\begin{aligned} s_R &:= \frac{4}{(h+2)} \sum_{\alpha \in \sqrt{2}\Phi^+(R)} w^-(\alpha), \\ t_R &:= \omega_R - s_R = \frac{2}{h+2} \omega_R + \frac{1}{h+2} \sum_{\alpha \in \sqrt{2}\Phi(R)} e^\alpha. \end{aligned} \quad (\text{A.3})$$

Then it is shown in [DLMN98] that  $s_R$  and  $t_R$  are mutually orthogonal Virasoro vectors. The central charge of  $t_R$  is  $2n/(n+3)$  if  $R$  is of type  $A_n$ , 1 if  $R$  is of type  $D_n$ , and  $6/7, 7/10$  and  $1/2$  if  $R$  is of type  $E_6, E_7$  and  $E_8$ , respectively. By Lemma A.2, the linear span of  $w^-(\alpha)$  with  $\alpha \in \sqrt{2}\Phi(R)$  forms a subalgebra of the Griess algebra of  $V_{\sqrt{2}R}^+$ . Indeed, this is the Griess algebra of the sub VOA generated by the Ising vectors  $w^-(\alpha)$ ,  $\alpha \in \sqrt{2}\Phi(R)$ . The Virasoro vector  $s_R$  provides its conformal vector and by the orthogonality we have the following (cf. Proposition 5.1 and Lemma 5.7 of [LSY07]).

**Proposition A.3** ([LSY07]).  $\langle w^-(\alpha) | \alpha \in \sqrt{2}\Phi(R) \rangle = \text{Com}_{V_{\sqrt{2}R}^+} \langle t_R \rangle$ .

The sub VOA  $\langle w^-(\alpha) | \alpha \in \sqrt{2}\Phi(R) \rangle$  is denoted by  $M_R$  and its automorphism group is determined in [LSY07].

**Remark A.4.** If  $R = E_8$ , then

$$t_{E_8} = \frac{1}{16}\omega_{E_8} + \frac{1}{32} \sum_{\alpha \in \sqrt{2}\Phi(E_8)} e^\alpha \quad (\text{A.4})$$

is an Ising vector. We say that  $t_{E_8}$  is the *standard Ising vector* of  $V_{\sqrt{2}E_8}^+$ . Recall that the dual lattice  $(\sqrt{2}E_8)^* = \frac{1}{\sqrt{2}}E_8$ . For  $x \in (\sqrt{2}E_8)^*$ , define a  $\mathbb{Z}$ -linear map

$$\begin{aligned} (x|\cdot) &: \sqrt{2}E_8 \longrightarrow \mathbb{Z}_2 \\ y &\longmapsto (x|y) \pmod{2}. \end{aligned}$$

Clearly the map

$$\begin{aligned} \varphi &: (\sqrt{2}E_8)^* \longrightarrow \text{Hom}_{\mathbb{Z}}(\sqrt{2}E_8, \mathbb{Z}_2) \\ x &\longmapsto (x|\cdot) \end{aligned}$$

is a group homomorphism and  $\text{Ker}\varphi = \sqrt{2}E_8$ . The map  $(x|\cdot)$  thus induces an automorphism  $\varphi_x$  of  $V_{\sqrt{2}E_8}$  given by

$$\varphi_x(u \otimes e^\alpha) = (-1)^{(x|\alpha)} u \otimes e^\alpha \quad \text{for } u \in M(1) \text{ and } \alpha \in \sqrt{2}E_8. \quad (\text{A.5})$$

In this case,

$$\varphi_x t_{E_8} = \frac{1}{16}\omega_{E_8} + \frac{1}{32} \sum_{\alpha \in \sqrt{2}\Phi(E_8)} (-1)^{(x|\alpha)} e^\alpha$$

is also an Ising vector. Since  $\varphi_x$  commutes  $\theta$  (the lift of the  $-1$  isometry), the Ising vector  $\varphi_x t_{E_8}$  is contained in  $V_{\sqrt{2}E_8}^+$ . The Ising vectors in  $V_{\sqrt{2}E_8}^+$  have been classified in [G98a, LSY07, LS07]. There are  $240 + 256 = 496$  Ising vectors and they can be divided into two different types:

**$A_1$ -type:**  $w^\pm(\alpha)$ ,  $\alpha \in \sqrt{2}E_8$ ,  $(\alpha|\alpha) = 4$  ( $|\Phi(E_8)| = 240$  Ising vectors),

**$E_8$ -type:**  $\varphi_x t_{E_8}$  for some  $x \in \frac{1}{\sqrt{2}}E_8$  ( $2^8 = 256$  Ising vectors).

**Remark A.5.** Let  $\alpha_1, \dots, \alpha_n$  be the simple roots of a root lattice of type  $A_n$ , that is,  $(\alpha_i|\alpha_i) = 2$ ,  $(\alpha_i|\alpha_j) = -1$  if  $|i - j| = 1$  and  $(\alpha_i|\alpha_j) = 0$  otherwise. Then

$$\text{Span}_{\mathbb{Z}}\{\alpha_1, \dots, \alpha_k\} \cong A_k$$

for any  $1 \leq k \leq n$  and we have a sequence of sublattices

$$A_1 \subset A_2 \subset \dots \subset A_n.$$

For  $1 \leq k \leq n$ , denote  $s^k = s_{A_k}$ . We also set

$$\eta^1 = s^1 \quad \text{and} \quad \eta^k = s^k - s^{k-1} \quad (\text{A.6})$$

for  $2 \leq k \leq n$ . It is shown in [DLMN98] (see also [GKO86]) that  $\eta^k$  is a simple Virasoro vector of central charge

$$c_k = 1 - \frac{6}{(k+2)(k+3)}, \quad 1 \leq k \leq n.$$

Moreover, we have the following orthogonal decomposition of the conformal vector of  $V_{\sqrt{2}A_n}$ .

$$\omega_{A_n} = \eta^1 + \eta^2 + \cdots + \eta^n + t_{A_n}. \quad (\text{A.7})$$

### A.3 Construction of $X^{[n]}$

Let  $n \geq 1$  and  $\epsilon_1, \dots, \epsilon_{n+5}$  an orthogonal basis of  $\mathbb{R}^n$  such that  $(\epsilon_i | \epsilon_j) = 2\delta_{i,j}$ . Set

$$K^{[k]} = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \cdots \oplus \mathbb{Z}\epsilon_k, \quad 1 \leq k \leq n+5.$$

Then  $K^{[k]} \cong A_1^{\oplus k}$ . Set  $\gamma = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$  and  $L = K^{[n+5]} \sqcup (K^{[n+5]} + \gamma)$ . Then  $L \cong D_4 \oplus A_1^{\oplus(n+5)}$ . We consider lattice VOAs

$$V_{K^{[1]}} \subset V_{K^{[2]}} \subset \cdots \subset V_{K^{[n+5]}} \subset V_L.$$

Note that  $V_{K^{[n+5]}}$  is the full sub VOA of  $V_L$ . We denote the conformal vector of  $V_L$  by  $\omega_L$ .

**Remark A.6.** Since  $(\alpha | \beta) \in 2\mathbb{Z}$  for any  $\alpha, \beta \in K^{[n+5]}$ , we can define  $V_{K^{[n+5]}}$  with a trivial 2-cocycle in  $Z^2(K^{[n+5]}, \{\pm 1\})$ . In the construction of  $V_L$ , we choose a 2-cocycle in  $Z^2(L, \{\pm 1\})$  such that its restriction on  $K^{[n+5]}$  is trivial. This is possible since we can form  $V_L = V_{K^{[n+5]}} \oplus V_{K^{[n+5]} + \gamma}$  as a  $\mathbb{Z}_2$ -graded simple current extension of  $V_{K^{[n+5]}}$ .

For  $1 \leq k \leq n+5$ , we set

$$\begin{aligned} H^k &= \epsilon_{1(-1)}^2 \mathbb{1} + \epsilon_{2(-1)}^2 \mathbb{1} + \cdots + \epsilon_{k(-1)}^2 \mathbb{1}, \\ E^k &= e^{\epsilon_1} + e^{\epsilon_2} + \cdots + e^{\epsilon_k}, \\ F^k &= e^{-\epsilon_1} + e^{-\epsilon_2} + \cdots + e^{-\epsilon_k}. \end{aligned} \quad (\text{A.8})$$

Then  $H^k, E^k, F^k$  form an  $\mathfrak{sl}_2$ -triple in the weight one subspace of  $V_{K^{[k]}}$  and generate a sub VOA isomorphic to level  $k$  affine VOA  $L_{\hat{\mathfrak{sl}}_2}(k, 0)$  associated to  $\hat{\mathfrak{sl}}_2$ . Let

$$\Omega^k = \frac{1}{4(k+2)} (H_{(-1)}^k H^k + 2E_{(-1)}^k F^k + 2F_{(-1)}^k E^k) \quad (\text{A.9})$$

be the Sugawara element. Then  $\Omega^k$  is the conformal vector of  $L_{\hat{\mathfrak{sl}}_2}(k, 0)$  (cf. [FZ92]). Consider the sublattice

$$M^{[k]} = \{\alpha \in K^{[k+1]} \mid (\alpha \mid \epsilon_1 + \cdots + \epsilon_{n+5}) = 0\}, \quad (\text{A.10})$$

Then  $M^{[k]} \cong \sqrt{2}A_k$  and we can define the Virasoro vectors

$$s^k = \frac{4}{k+3} \sum_{1 \leq i < j \leq k+1} w^-(\epsilon_i - \epsilon_j), \quad \eta^1 = w^-(\epsilon_1 - \epsilon_2), \quad \eta^k = s^k - s^{k-1} \quad (\text{A.11})$$

for  $2 \leq k \leq n+4$  as in (A.3) and (A.6). It is shown in [GKO86, LY04] that the Virasoro vectors  $\eta^k$  can be written as

$$\eta^k = \Omega^k + \frac{1}{4} \epsilon_{k+1(-1)}^2 \mathbb{1} - \Omega^{k+1}.$$

Therefore, we have the following orthogonal decompositions.

$$\omega_L = s^{n+4} \dot{+} \Omega^{n+5} = \eta^1 \dot{+} \eta^2 \dot{+} \cdots \dot{+} \eta^{n+4} \dot{+} \Omega^{n+5}. \quad (\text{A.12})$$

As a consequence, we have the following full sub VOA of  $V_L$ .

$$L(c_1, 0) \otimes L(c_2, 0) \otimes \cdots \otimes L(c_{n+4}, 0) \otimes L_{\hat{\mathfrak{sl}}_2}(n+5, 0). \quad (\text{A.13})$$

We note that all  $\eta^k$ ,  $1 \leq k \leq n+4$ , and  $\Omega^{n+5}$  are defined inside the compact form  $V_{L, \mathbb{R}}^+$ .

**Lemma A.7.** *The weight one subspace of  $\text{Com}_{V_L} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$  is trivial.*

**Proof:** It suffices to show  $\text{Ker}_{V_L} \eta^1_{(1)} \cap \text{Ker}_{V_L} \eta^2_{(1)} \cap \text{Ker}_{V_L} \Omega^{n+5}_{(1)} \cap (V_L)_1 = 0$ . Since  $V_L = V_{K[n+5]} \oplus V_{K[n+5]+\gamma}$ , we have a decomposition  $(V_L)_1 = (V_{K[n+5]})_1 \oplus (V_{K[n+5]+\gamma})_1$ . The weight one subspace of  $V_{K[n+5]}$  is  $3(n+5)$ -dimensional spanned by  $\epsilon_{i(-1)} \mathbb{1}$  and  $e^{\pm \epsilon_i}$  for  $1 \leq i \leq n+5$ . By a direct computation one has

$$\begin{aligned} \Omega^{n+5}_{(1)} \epsilon_{i(-1)} \mathbb{1} &= \frac{2}{n+7} \epsilon_{i(-1)} \mathbb{1} + \frac{1}{n+7} H^{n+5}, \\ \Omega^{n+5}_{(1)} e^{\epsilon_i} &= \frac{2}{n+7} e^{\epsilon_i} + \frac{1}{n+7} E^{n+5}, \\ \Omega^{n+5}_{(1)} e^{-\epsilon_i} &= \frac{2}{n+7} e^{-\epsilon_i} + \frac{1}{n+7} F^{n+5}. \end{aligned}$$

Therefore, the characteristic polynomial of  $\Omega^{n+5}_{(1)}$  on  $(V_{K[n+5]})_1$  is

$$(x-1)^3 \left( x - \frac{2}{n+7} \right)^{3n+12}$$

and hence  $\text{Ker}_{V_L} \Omega^{n+5}_{(1)} \cap (V_{K[n+5]})_1 = 0$ . The weight one subspace of  $V_{K[n+5]+\gamma}$  is spanned by the vectors  $e^\alpha$  with  $\alpha = \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ . Set  $\gamma_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$ ,  $\gamma_2 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$ ,  $\gamma_3 = \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)$  and  $\tilde{e}(\gamma_i) = e^{\gamma_i} + e^{-\gamma_i}$  for  $1 \leq i \leq 3$ . Then by a direct computation one has

$$\begin{aligned} \text{Ker}_{V_L} \Omega^{n+5}_{(1)} \cap (V_{K[n+5]+\gamma})_1 &= \text{Span}\{ \tilde{e}(\gamma_1) - \tilde{e}(\gamma_2), \tilde{e}(\gamma_2) - \tilde{e}(\gamma_3) \}, \\ \text{Ker}_{V_L} \eta^1_{(1)} \cap \text{Ker}_{V_L} \Omega^{n+5}_{(1)} \cap (V_{K[n+5]+\gamma})_1 &= \text{Span}\{ 2\tilde{e}(\gamma_1) - \tilde{e}(\gamma_2) - \tilde{e}(\gamma_3) \}, \\ \eta^2_{(1)} (2\tilde{e}(\gamma_1) - \tilde{e}(\gamma_2) - \tilde{e}(\gamma_3)) &= \frac{3}{5} (2\tilde{e}(\gamma_1) - \tilde{e}(\gamma_2) - \tilde{e}(\gamma_3)). \end{aligned}$$

Therefore,  $\text{Ker}_{V_L} \eta^1_{(1)} \cap \text{Ker}_{V_L} \eta^2_{(1)} \cap \text{Ker}_{V_L} \Omega^{n+5}_{(1)} \cap (V_{K^{[n+5]+\gamma}})_1 = 0$  and thus the weight one subspace of  $\text{Com}_{V_L} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$  is trivial.  $\blacksquare$

**Corollary A.8.** *The subalgebra  $\text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$  is a compact VOA of OZ-type.*

We will realize the (2A,3A)-generated subalgebra  $X^{[n]} = \langle a, b, x^1, \dots, x^n \rangle$  in Section 4.3 as a subalgebra of  $\text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$ . Set

$$\begin{aligned} q &:= \sum_{k=1}^3 (2\gamma + \epsilon_k - \epsilon_4 - 4\epsilon_5)_{(-1)} (e^{\gamma - \epsilon_k - \epsilon_4} - e^{-\gamma + \epsilon_k + \epsilon_4}) \\ &\quad - 4 \sum_{k=1}^3 (e^{\gamma - \epsilon_k - \epsilon_5} + e^{-\gamma + \epsilon_k + \epsilon_5}) + 12 (e^{\gamma - \epsilon_4 - \epsilon_5} + e^{-\gamma + \epsilon_4 + \epsilon_5}) \in V_{L,\mathbb{R}}^+. \end{aligned} \quad (\text{A.14})$$

The following is shown in [SY03].

**Lemma A.9** ([SY03]). *The vector  $q$  is a highest weight vectors for  $\eta^1, \eta^2, \eta^3, \eta^4, \Omega^{n+5}$  and  $w^-(\epsilon_4 - \epsilon_5)$  with highest weights  $0, 0, 2/3, 4/3, 0$  and  $1/16$ , respectively. In particular,  $q \in \text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$ .*

We set

$$a := w^-(\epsilon_4 - \epsilon_5), \quad b := -\frac{1}{2}a + \frac{15}{64}\eta^3 + \frac{21}{32}\eta^4 + \frac{\sqrt{3}}{2^7}q, \quad c := \tau_a b. \quad (\text{A.15})$$

Then it is shown in [SY03] that  $a$  and  $b$  are Ising vectors in  $V_L$  such that  $\langle a, b \rangle$  is isomorphic to the 3A-algebra with the characteristic Virasoro frame  $\eta^3 \dot{+} \eta^4$ . For  $1 \leq i \leq n$ , we set

$$x^i := w^-(\epsilon_5 - \epsilon_{i+5}). \quad (\text{A.16})$$

Using Lemma A.2 we can verify the following.

**Lemma A.10.**  *$(a | x^i) = (b | x^i) = (x^j | x^k) = 2^{-5}$  for  $1 \leq i \leq n$  and  $1 \leq j < k \leq n$ .*

**Lemma A.11.**  *$\langle a, b, x^1, \dots, x^n \rangle$  is a subalgebra of  $\text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$ .*

**Proof:** It follows from Proposition A.3 and Eq. (A.12) that  $a = w^-(\epsilon_4 - \epsilon_5)$  and  $x^i = w^-(\epsilon_5 - \epsilon_{i+5})$ ,  $1 \leq i \leq n$ , belong to the commutant of  $\langle \Omega^{n+5} \rangle$ . It also follows from Proposition A.3 that  $s^2 = \eta^1 \dot{+} \eta^2$  is the conformal vector of  $\langle w^-(\epsilon_1 - \epsilon_2), w^-(\epsilon_2 - \epsilon_3) \rangle$ . By Lemma A.2,  $a$  and  $x^i$ ,  $1 \leq i \leq n$ , are orthogonal to both of  $w^-(\epsilon_1 - \epsilon_2)$  and  $w^-(\epsilon_2 - \epsilon_3)$ . Therefore  $\langle a, x^1, \dots, x^n \rangle \subset \text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$ . That  $b \in \text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$  follows from Lemma A.9. Hence,  $\langle a, b, x^1, \dots, x^n \rangle$  is a subalgebra of  $\text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$ .  $\blacksquare$



**Remark A.12.** The Ising vectors in (A.15) and (A.16) are defined inside  $V_{L,\mathbb{R}}^+$  so that  $\langle a, b, x^1, \dots, x^n \rangle$  has a compact real form as a subalgebra of  $\text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$  which is of OZ-type by Corollary A.8. Since  $\langle a, b \rangle$  is the 3A-type and all  $\langle a, x^i \rangle$ ,  $\langle b, x^i \rangle$  and  $\langle x^j, x^k \rangle$  are the 2A-type by Lemma A.10, we have obtained a realization of the (2A,3A)-generated subalgebra  $X^{[n]} = \langle a, b, x^1, \dots, x^n \rangle$  discussed in Section 4.3 inside  $V_{L,\mathbb{R}}^+$ .

**Proposition A.13.**  $\langle a, b, x^1, \dots, x^n \rangle$  has the Virasoro frame  $\eta^3 \dot{+} \eta^4 \dot{+} \dots \dot{+} \eta^{n+4}$ . In particular,  $\langle a, b, x^1, \dots, x^n \rangle$  is a full subalgebra of  $\text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$ .

**Proof:** Since  $\langle a, b \rangle$  is isomorphic to the 3A-algebra with Virasoro frame  $\eta^3 \dot{+} \eta^4$ , we have

$$a + b + c = \frac{15}{32}\eta^3 + \frac{21}{16}\eta^4.$$

By (A.11) we have

$$\begin{aligned} a + b + c &= \frac{15}{32}\eta^3 + \frac{21}{16}\eta^4 = \frac{15}{32}(s^3 - s^2) + \frac{21}{16}(s^4 - s^3) \\ &= -\frac{15}{32}s^2 + \frac{3}{16} \sum_{1 \leq i < j \leq 4} w^-(\epsilon_i - \epsilon_j) + \frac{3}{4} \sum_{1 \leq i \leq 4} w^-(\epsilon_i - \epsilon_5). \end{aligned} \quad (\text{A.17})$$

It is clear from expressions that  $s^2$  and  $w^-(\epsilon_i - \epsilon_j)$  are orthogonal to  $w^-(\epsilon_5 - \epsilon_{k+5})$  for  $1 \leq i < j \leq 4$  and  $1 \leq k \leq n$ , whereas  $(w^-(\epsilon_i - \epsilon_5) | w^-(\epsilon_5 - \epsilon_{k+5})) = 2^{-5}$  and one has  $w^-(\epsilon_i - \epsilon_5) \circ w^-(\epsilon_5 - \epsilon_{k+5}) = w^-(\epsilon_i - \epsilon_{k+5})$  by Lemma A.2. Therefore we have

$$(a + b + c) \circ x^k = -\frac{15}{32}s^2 + \frac{3}{16} \sum_{1 \leq i < j \leq 4} w^-(\epsilon_i - \epsilon_j) + \frac{3}{4} \sum_{1 \leq i \leq 4} w^-(\epsilon_i - \epsilon_{k+5}). \quad (\text{A.18})$$

So we have

$$\begin{aligned} &a + b + c + \sum_{k=1}^n (a + b + c) \circ x^k \\ &= -\frac{15(n+1)}{32}s^2 - \frac{3(n+1)}{16} \sum_{1 \leq i < j \leq 4} w^-(\epsilon_i - \epsilon_j) + \frac{3}{4} \sum_{\substack{1 \leq i \leq 4 \\ 5 \leq k \leq n+5}} w^-(\epsilon_i - \epsilon_k). \end{aligned}$$

Now we consider the conformal vector  $\omega^n$  of  $\langle a, b, x^1, \dots, x^n \rangle$  in (4.13).

$$\begin{aligned}
\omega^n &= \frac{3(3-n)}{2(n+7)}\eta^3 + \frac{16}{3(n+7)} \left( a + b + c + \sum_{i=1}^n (a + b + c) \circ x^i \right) \\
&\quad + \frac{4}{n+7} \left( \sum_{i=1}^n x^i + \sum_{1 \leq j < k \leq n} x^j \circ x^k \right) \\
&= \frac{3(3-n)}{2(n+7)} \left( \frac{2}{3} \sum_{1 \leq i \leq 4} w^-(\epsilon_i - \epsilon_j) - s^2 \right) \\
&\quad + \frac{16}{3(n+7)} \left( -\frac{15(n+1)}{32} s^2 + \frac{3(n+1)}{16} \sum_{1 \leq i < j \leq 4} w^-(\epsilon_i - \epsilon_j) + \frac{3}{4} \sum_{\substack{1 \leq i \leq 4 \\ 5 \leq k \leq n+5}} w^-(\epsilon_i - \epsilon_k) \right) \\
&\quad + \frac{4}{n+7} \left( \sum_{i=1}^n w^-(\epsilon_5 - \epsilon_{i+5}) + \sum_{1 \leq j < k \leq n} w^-(\epsilon_{j+5} - \epsilon_{k+5}) \right) \\
&= -s^2 + \frac{4}{n+7} \sum_{1 \leq i < j \leq n+5} w^-(\epsilon_i - \epsilon_j) = s^{n+4} - s^2.
\end{aligned}$$

Thus, we have

$$\omega^n = s^{n+4} - s^2 = (s^3 - s^2) + (s^4 - s^3) + \dots + (s^{n+5} - s^{n+4}) = \eta^3 \dot{+} \eta^4 \dot{+} \dots \dot{+} \eta^{n+4}.$$

Since  $\omega^k$  is the conformal vector of a subalgebra  $\langle a, b, x^1, \dots, x^k \rangle$  of  $\langle a, b, x^1, \dots, x^n \rangle$  for  $1 \leq k \leq n$ , the Virasoro vector  $\eta^{k+4} = \omega^k - \omega^{k-1}$  belongs to  $\langle a, b, x^1, \dots, x^n \rangle$ . Therefore,  $\langle a, b, x^1, \dots, x^n \rangle$  has a Virasoro frame  $\omega^n = \eta^3 \dot{+} \eta^4 \dot{+} \dots \dot{+} \eta^n$ .  $\blacksquare$

Denote  $Y^{[0]} = \langle a, b \rangle \subset V_{L, \mathbb{R}}^+$  and  $Y^{[k]} = \langle a, b, x^1, \dots, x^k \rangle \subset V_{L, \mathbb{R}}^+$  for  $1 \leq k \leq n$ . Then  $Y^{[k]}$  provides a realization of  $X^{[k]}$  defined in (4.12). Using the Virasoro frame  $\eta^3 \dot{+} \eta^4 \dot{+} \dots \dot{+} \eta^{n+4}$  we can prove that the commutant  $\text{Com}_{Y^{[n]}} Y^{[n-1]}$  is generated by the  $c = c_{n+4}$  Virasoro vector  $\eta^{n+4} = \omega^n - \omega^{n-1}$  which is denoted by  $f^n$  in Theorem 4.18. Consider the full subalgebra of  $V_L$  displayed in (A.13). The irreducible decomposition with respect to this subalgebra is known as the GKO construction. By Eq. (2.20) of [GKO86] (see also Lemma 3.1 of [LLY03]), we have the following decompositions:

$$\begin{aligned}
V_{K^{[n+5]}} &= \bigoplus_{\substack{1 \leq i_k \leq k+1 \\ i_k \equiv 1 \pmod{2}}} L(c_1, h_{i_1, i_2}^{(1)}) \otimes \dots \otimes L(c_{n+4}, h_{i_{n+4}, i_{n+5}}^{(n+4)}) \otimes L_{\hat{\mathfrak{sl}}_2}(n+5, i_{n+5} - 1), \\
V_{K^{[n+5]+\gamma}} &= \bigoplus_{\substack{1 \leq j_k \leq k+1 \\ j_k \equiv 1 + \delta_k \pmod{2}}} L(c_1, h_{j_1, j_2}^{(1)}) \otimes \dots \otimes L(c_{n+4}, h_{j_{n+4}, j_{n+5}}^{(n+4)}) \otimes L_{\hat{\mathfrak{sl}}_2}(n+5, j_{n+5} - 1),
\end{aligned} \tag{A.19}$$

where  $\delta_1 = 1$ ,  $\delta_2 = 0$ ,  $\delta_3 = 1$  and  $\delta_k = 0$  for  $4 \leq k \leq n+5$ . By the decompositions above, we see that the commutant of  $\langle \eta^1, \eta^2, \dots, \eta^{n+3}, \Omega^{n+5} \rangle$  in  $V_{L, \mathbb{R}}$  is exactly  $\langle \eta^{n+4} \rangle$ . Therefore,  $\text{Com}_{Y^{[n]}} Y^{[n-1]}$  is generated by the  $c = c_{n+4}$  Virasoro vector  $\eta^{n+4} = \omega^n - \omega^{n-1}$ .

**Conjecture A.14.** The subalgebra  $X^{[n]} = \langle a, b, x^1, \dots, x^n \rangle$  defined in (4.12) is unique up to isomorphism and coincides with the subalgebra  $\text{Com}_{V_{L,\mathbb{R}}^+} \langle \eta^1, \eta^2, \Omega^{n+5} \rangle$  of  $V_{L,\mathbb{R}}^+$ . In particular,  $\text{Com}_{X^{[n]}} X^{[n-1]}$  is generated by the  $c = c_{n+4}$  Virasoro vector  $f^n$  in Theorem 4.18.

Finally, we prove that the Virasoro vectors in (4.15) is linearly independent. By (A.16) and (A.17), we know that the following vectors are linearly independent in  $V_{K[n+5]}$ .

$$\begin{aligned} u_{a,b} &= \eta^3, \quad x^i = w^-(\epsilon_5 - \epsilon_{i+5}), \quad a = w^-(\epsilon_4 - \epsilon_5), \quad a \circ x^i = w^-(\epsilon_4 - \epsilon_{i+5}), \\ a + b + c, \quad (a + b + c) \circ x^i, \quad x^j \circ x^k &= w^-(\epsilon_{j+5} - \epsilon_{k+5}), \quad 1 \leq i \leq n, \quad 1 \leq j < k \leq n. \end{aligned}$$

On the other hand, by (A.14) and (A.15) we have

$$\begin{aligned} q = \frac{2^6}{\sqrt{3}}(b - c) &= \sum_{k=1}^3 (2\gamma + \epsilon_k - \epsilon_4 - 4\epsilon_5)_{(-1)} (e^{\gamma - \epsilon_k - \epsilon_4} - e^{-\gamma + \epsilon_k + \epsilon_4}) \\ &\quad - 4 \sum_{k=1}^3 (e^{\gamma - \epsilon_k - \epsilon_5} + e^{-\gamma + \epsilon_k + \epsilon_5}) + 12 (e^{\gamma - \epsilon_4 - \epsilon_5} + e^{-\gamma + \epsilon_4 + \epsilon_5}). \end{aligned}$$

Since  $\sigma_{x^i}$  acts on  $V_L$  by a reflection associated to  $\epsilon_5 - \epsilon_{i+5}$ , we have

$$\begin{aligned} \frac{2^6}{\sqrt{3}}(b - c) \circ x^i &= \sum_{k=1}^3 (2\gamma + \epsilon_k - \epsilon_4 - 4\epsilon_{i+5})_{(-1)} (e^{\gamma - \epsilon_k - \epsilon_4} - e^{-\gamma + \epsilon_k + \epsilon_4}) \\ &\quad - 4 \sum_{k=1}^3 (e^{\gamma - \epsilon_k - \epsilon_{i+5}} + e^{-\gamma + \epsilon_k + \epsilon_{i+5}}) + 12 (e^{\gamma - \epsilon_4 - \epsilon_{i+5}} + e^{-\gamma + \epsilon_4 + \epsilon_{i+5}}). \end{aligned}$$

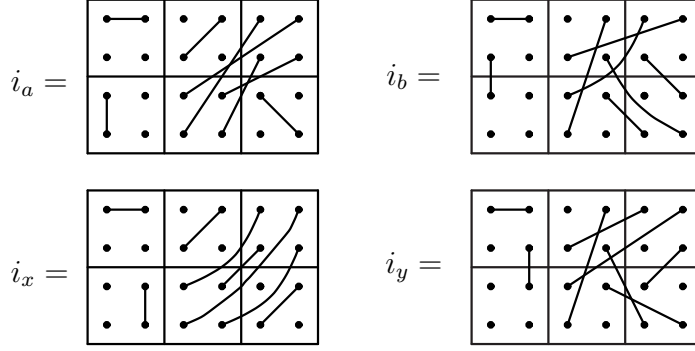
From the expressions above, we see that  $b - c$  and  $(b - c) \circ x^i$ ,  $1 \leq i \leq n$ , are linearly independent in  $V_{K[n+5]+\gamma}$ . Thus, all the Virasoro vectors in (4.15) are linearly independent.

#### A.4 Construction of $\langle a, b, x, y \rangle$ with $\langle x, y \rangle \cong U_{3A}$

In this subsection, we will construct explicitly four Ising vectors  $a, b, x, y$  in the VOA  $V_\Lambda^+$  associated to the Leech lattice  $\Lambda$  such that  $\langle a, b \rangle \cong \langle x, y \rangle \cong U_{3A}$  and  $\langle a, x \rangle \cong \langle b, x \rangle \cong \langle a, y \rangle \cong \langle b, y \rangle \cong U_{2A}$ . Therefore, the Griess algebra discussed in Section 4.2 does exist.

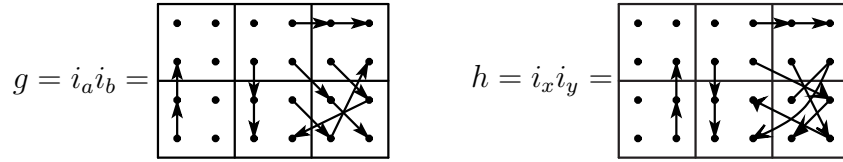
For explicit calculations, we will use the notion of *hexacode balance* (or MOG) to denote the codewords of the Golay code  $\mathcal{G}$  and the vectors in the Leech lattice  $\Lambda$  [CS99, G98b]. Namely, we arrange the set  $\Omega = \{1, 2, \dots, 24\}$  into a  $4 \times 6$  array such that the six columns forms a sextet. Recall that the Conway group  $\text{Co}_0$  is the orthogonal group  $\text{O}(\Lambda)$  of  $\Lambda$  and contains 4 conjugacy classes of involutions (see [CS99]). If  $i$  is an involution with trace 8 on  $\Lambda$ , then the  $(-1)$ -eigenlattice is isomorphic to  $\sqrt{2}E_8$ .

Consider the following 4 involutions in  $\text{O}(\Lambda)$ .



The automorphisms above should be viewed as permutations on 24 coordinates and they act on the Leech lattice  $\Lambda$  from the left.

Note that



where a composition of above permutations is executed from right to left.

We have  $\langle i_a, i_b \rangle \cong \langle i_x, i_y \rangle \cong S_3$ . Note that  $i_x$  and  $i_y$  commute with  $\langle i_a, i_b \rangle$ . Thus, we also have  $\langle i_a, i_b, i_x \rangle \cong \langle i_a, i_b, i_y \rangle \cong \text{Dih}_{12}$ , a dihedral group of order 12.

**Notation A.15.** Let  $A, B, X, Y$  be the  $(-1)$ -eigenlattices of  $i_a, i_b, i_x$  and  $i_y$ , respectively. Then  $A \cong B \cong X \cong Y \cong \sqrt{2}E_8$ . Moreover,  $A + B \cong X + Y \cong \text{DIH}_6(14)$  and  $A + B + X \cong A + B + Y \cong \text{DIH}_{12}(16)$  by the analysis in [GL11].

**Notation A.16** (cf. Remark A.4). For any  $\sqrt{2}E_8$ -sublattice  $M$ , let

$$t_M = \frac{1}{16}\omega_M + \frac{1}{32} \sum_{\substack{\alpha \in M \\ (\alpha|\alpha)=4}} e^\alpha$$

be the standard Ising vector in  $V_M^+$ .

Since  $A$  and  $B$  are doubly even, it is possible to choose a 2-cocycle  $\varepsilon \in \mathbb{Z}^2(\Lambda, \{\pm 1\})$  such that  $\varepsilon$  is trivial on  $A$  and  $B$  (see [LY14, Notation 5.38]). Let  $a := t_A$  and  $b := t_B$  be the standard Ising vector in  $V_A^+$  and  $V_B^+$ , respectively.

**Lemma A.17.** *Let  $M, N, E$  be  $\sqrt{2}E_8$ -sublattices of the Leech lattice. Suppose  $M + N \cong \text{DIH}_6(14)$  and  $M + E \cong N + E \cong \text{DIH}_4(12)$ . Then there exists an Ising vector  $e \in V_E^+$  such that  $(t_M | e) = (t_N | e) = 2^{-5}$ .*

**Proof:** Let  $i_L$  be the SSD involution associated to a  $\sqrt{2}E_8$ -sublattice  $L$ . Since  $i_E i_N$  has order 2 and trace 8, the  $(-1)$ -eigenlattice of  $i_E i_N$  is isometric to  $\sqrt{2}E_8$ . Let  $N'$  be the  $(-1)$ -eigenlattice of  $i_E i_N$ . Then  $\langle i_M, i_N i_E \rangle \cong \text{Dih}_{12}$  and hence  $M+N' \cong M+N+E \cong \text{DIH}_{12}(16)$  as defined in [GL11]. Let  $e'$  be an Ising vector of  $E_8$ -type in  $V_{N'}^+$  (see Remark A.4 for definition). Then  $\langle t_M, e' \rangle \cong U_{6A}$  since  $\tau_{t_M} \tau_{e'}$  has order 6.

Let  $\tilde{e}$  be the central Ising vector in  $\langle t_M, e' \rangle \cong U_{6A}$  (cf. Eq. (2.9)). Then  $\tilde{e} \in V_E^+$ . Let  $\tilde{t}_N = e' \circ t_N$ . Then  $\tilde{t}_N \in V_{N'}^+$  and

$$(t_M | \tilde{e}) = (\tilde{t}_N | \tilde{e}) = \frac{1}{32} \quad \text{and} \quad (t_M | \tilde{t}_N) = \frac{13}{2^{10}}. \quad (\text{A.20})$$

By the classification of Ising vectors in [LSY07] and [LS07] (cf. Remark A.4),  $\tilde{t}_N = \varphi_x(t_N)$  for some  $x \in N^*$ . Moreover, by Proposition 3.4 of [GL12b], there exists an  $\alpha \in M$  such that  $\varphi_\alpha|_{V_N^+} = \varphi_x$ . Hence, we have

$$\varphi_\alpha(t_M) = t_M \quad \text{and} \quad \varphi_\alpha(\tilde{t}_N) = \varphi_\alpha \varphi_x(t_N) = t_N.$$

Now let  $e = \varphi_\alpha(\tilde{e})$ . Then we have

$$(t_M | e) = (t_N | e) = \frac{1}{32}$$

by Equation (A.20). ■

Finally, by Lemma A.17, there exist Ising vectors  $x \in V_X^+$  and  $y \in V_Y^+$  such that  $\langle a, x \rangle \cong \langle b, x \rangle \cong \langle a, y \rangle \cong \langle b, y \rangle \cong U_{2A}$ . Since  $X+Y \cong \text{DIH}_6(14)$ , we have  $(x | y) = 13 \cdot 2^{-10}$  or  $5 \cdot 2^{-10}$ . By Theorem 3.5, the case  $(x | y) = 5 \cdot 2^{-10}$  is impossible. Hence we have  $(x | y) = 13 \cdot 2^{-10}$  and  $\langle x, y \rangle \cong U_{3A}$  as desired.

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